Differential Geometry 230ar

Homework 2

Due: Monday October 25th

Please attempt all of the problems. But you only need to hand in five of them, and you'll receive full credit if you answer all five correctly.

1. Recall that on a smooth Riemannian manifold (M, g), the Riemannian curvature is given by

$$R^k{}_{lij} = \partial_i \Gamma^k_{jl} - \partial_j \Gamma^k_{il} + \Gamma^k_{ip}\Gamma^p_{jl} - \Gamma^k_{jp}\Gamma^p_{il},$$

where Γ^k_{jl} are the Christoffel symbols for g. Show that for a covector field $a = a_i dx^i$,

$$(\nabla_i \nabla_j - \nabla_j \nabla_i) a_k = -R^l{}_{kij} a_l.$$

2. We define $R_{hkij} = g_{hl} R^l{}_{kij}$. We also define the map R acting on vector fields W, Z, X and Y by

$$R(W, Z, X, Y) = g(W, \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z).$$

   (a) Show that $R(\partial_h, \partial_k, \partial_i, \partial_j) = R_{hkij}$.

   (b) Show that R is a covariant tensor of order 4. That is, for functions f_1, f_2, f_3, f_4 in $C^\infty(M)$,

$$R(f_1 W, f_2 Z, f_3 X, f_4 Y) = f_1 f_2 f_3 f_4 R(W, Z, X, Y).$$

3. Show that

$$\begin{aligned} R_{hkij} &= \frac{1}{2}\left(\partial_k \partial_j g_{hi} + \partial_h \partial_i g_{kj} - \partial_h \partial_j g_{ki} - \partial_k \partial_i g_{hj}\right) \\ &\quad + g_{pq}\left(\Gamma^p_{hi}\Gamma^q_{kj} - \Gamma^p_{hj}\Gamma^q_{ki}\right). \end{aligned}$$

4. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be a smooth function. The graph of f given by

$$G(f) = \left\{(x^1, x^2, f(x^1, x^2)) \in \mathbf{R}^3 \mid x^1, x^2 \in \mathbf{R}\right\}$$

is a smooth two dimensional manifold with a single coordinate chart ϕ given by

$$\phi(x^1, x^2, f(x^1, x^2)) = (x^1, x^2).$$

Show that the metric on $G(f)$ obtained from pulling back the standard metric on $\mathbf{R}^3$ is written in this coordinate system as

$$g_{ij} = \delta_{ij} + \frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}.$$

5. Recall that the Ricci curvature is given by

$$R_{lj} = g^{hi} R_{hlij},$$

and that the scalar curvature is the function given by

$$R = g^{lj} R_{lj}.$$

Calculate the scalar curvature of (S^2, g), where g is the metric on S^2 obtained from pulling back the metric on $\mathbf{R}^3$.

6. We showed in class that for $t \in [0, \infty)$, if $f_t = f(x,t)$ are smooth functions on $\mathbf{R}$ with the property that there exist constants A_k (independent of t) with

$$\|f_t\|_{C^k} \leq A_k,$$

then there exists a smooth function f_∞ and a sequence of times t_i such that f_{t_i} converges to f_∞ in C^∞ on compact subsets. Show by counterexample that the phrase 'on compact subsets' is needed for this statement to be true: specifically, find $f_t = f(x,t)$ satisfying the above estimates for some A_k but such that there does not exist any function f_∞ and any sequence of times t_i with

$$\|f_{t_i} - f_\infty\|_{C^0} \to 0.$$

7. The Harnack inequality for positive solutions $h(x,t)$ of the heat equation on $\mathbf{R}$ is

$$h(x_2, t_2) \geq \sqrt{\frac{t_1}{t_2}} e^{-(x_2-x_1)^2/4(t_2-t_1)} h(x_1, t_1), \tag{1}$$

for any (x_1, t_1), (x_2, t_2) satisfying $0 < t_1 < t_2$. The *fundamental solution* ρ_y with center $y \in \mathbf{R}$ of the heat equation is given by

$$\rho_y(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-(x-y)^2/4t}.$$

Show that the Harnack inequality is sharp by finding $y \in \mathbf{R}$ depending on (x_1, t_1) and (x_2, t_2) such that equality holds in (1) for ρ_y.

8. Suppose that $f = f(x,t)$ satisfies the 'endangered species equation'

$$f_t = \partial_x^2 f + f^2,$$

for $-\infty < x < \infty$, $t \geq 0$. Suppose also that $f > 0$. Then show that for $t > 0$ we have the differential Harnack inequality

$$\partial_t f + \frac{2f}{3t} \geq \frac{(\partial_x f)^2}{f} + \frac{1}{2} f^2.$$

(Hint: Try the Harnack quantity $H = \partial_x^2 l + \alpha(\partial_x l)^2 + \beta e^l + \phi$, where $l = \log f$, α and β are constants and ϕ is a function of x and t to be determined.)