**Differential Geometry 230ar**

**Homework 3**

**Due: Monday November 8th**

Please attempt all of the problems. But you only need to hand in five of them, and you'll receive full credit if you answer all five correctly.

Unless otherwise stated, assume that we are working on a Riemannian manifold $(M, g)$ of dimension $n$.

1. Give a proof of the second Bianchi identity,
$$\nabla_h R_{ijkl} + \nabla_k R_{ijlh} + \nabla_l R_{ijhk} = 0,$$
by calculating in a normal coordinate system.

2. Fix a point $p$ in a Riemannian manifold $(M, g)$. The exponential map $\exp_p$ given in class defines a diffeomorphism from an open neighbourhood $\Omega$ of the origin in $T_pM$ to $\exp_p(\Omega)$. Choose an orthonormal frame $F_1, \ldots, F_n$ for $T_pM$ with respect to the metric $g_p$. Let $\phi$ be the coordinate map on $\Omega$ sending a vector $X = X^i F_i$ to $(X^1, \ldots, X^n) \in \mathbf{R}^n$. Show that the chart $(\exp_p(\Omega), \phi \circ \exp_p^{-1})$ gives normal coordinates $y^i$ at $p$ with the property that geodesics through $p$ are described by the straight lines through the origin $y^i = a^i t$ for constants $a^i$.

3. Let $f$ be a smooth function on $M$. Recall that the Laplacian of $f$ is given by $\triangle f = g^{ij} \nabla_i \nabla_j f$. Show that
$$g^{ij} g^{pq} \nabla_i \nabla_p f \nabla_j \nabla_q f \geq \frac{1}{n} (\triangle f)^2.$$
(Optional: is this inequality sharp?)

4. In class we proved the Li-Yau Harnack inequality for *positive* solutions of the heat equation. That is, for $h = h(x, t) > 0$ satisfying the heat equation $\partial_t h = \triangle h$ on the compact Riemannian manifold $(M, g)$ with $R_{ij} \geq -K g_{ij}$ $(K \geq 0)$ and for any $(x_1, t_1)$, $(x_2, t_2)$ in $M \times [0, \infty)$ with $0 < t_1 < t_2$, and for any $\alpha > 1$,
$$h(x_2, t_2) \geq \left( \frac{t_1}{t_2} \right)^{\frac{n\alpha}{2}} e^{-\frac{\alpha d^2(x_1, x_2)}{4(t_2 - t_1)} - \frac{n}{2} \alpha (\alpha - 1)^{-1} K (t_2 - t_1)} h(x_1, t_1).$$

Show that the same inequality holds if we assume only that $h = h(x,t)$ is *nonnegative*. (Hint: consider $l = \log(h+\epsilon)$ and let $\epsilon \to 0$. You don't need to reproduce all the calculations. Just outline the main steps of the proof and explain why they work with the new $l$.)

5. In class we showed that if $h = h(x,t)$ is a solution of the heat equation on the compact Riemannian manifold $(M,g)$ then there exist constants $A_0$ and $A_1$ depending only on $h(x,0)$ and $(M,g)$ such that

$$\|h\|_{C^0} \le A_0 \quad \text{and } \|h\|_{C^1} \le A_1.$$

Using a similar maximum principle argument show that there exists $A_2$ depending only on $h(x,0)$ and $(M,g)$ such that

$$\|h\|_{C^2} \le A_2.$$

6. Use induction to show that there exist constants $A_k$ for $k = 0,1,2,\dots$ depending only on $h(x,0)$ and $(M,g)$ such that

$$\|h\|_{C^k} \le A_k.$$

7. Hamilton's Ricci flow is a flow of metrics $g = g(x,t)$ on a manifold $M$, given by

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}.$$

(a) Show that if $M$ has dimension 2,

$$R_{ij} = \frac{1}{2} R g_{ij}.$$

(b) Let $M = S^2$ and let $g(x,0)$ be the standard metric on $S^2$ induced from the metric on $\mathbf{R}^3$. Describe the behavior of the Ricci flow in this case.