Differential Geometry 230ar

Homework 4

Due: Monday November 22nd

Please attempt all of the problems. But you only need to hand in five of them, and you'll receive full credit if you answer all five correctly.

1. Let (M, g) be a compact smooth Riemannian manifold, and let $\triangle$ be the Laplacian associated to g. Show that

   (a) The operator $P = \dfrac{\partial}{\partial t} + \triangle^2$ is parabolic.

   (b) The integrals

   $$\int_M h^2 d\mu_g \quad \text{and} \quad \int_M |\nabla h|^2 d\mu_g$$

   are bounded along the flow (where $d\mu_g$ is the usual volume form associated to g and h is a solution of $Ph = 0$.)

   (Recall the definition of a parabolic operator: Let L be a differential operator of order $2l$ acting on functions on a manifold M. The *symbol* $\sigma_{L,p}$ of L at a point p acts on cotangent vectors $\xi \in T_pM^*$ by

   $$\sigma_{L,p}(\xi) = L(\phi^{2l})(p),$$

   where ϕ is a function defined in a neighborhood of p satisfying $\partial_i \phi = \xi_i$. We then say that L is *strongly elliptic* at a point p if there exists $c > 0$ such that

   $$(-1)^l \sigma_{L,p}(\xi) \geq c|\xi|^{2l},$$

   for all cotangent vectors ξ. We say that an operator P is *parabolic* if it is of the form

   $$P = \frac{\partial}{\partial t} + L,$$

   where L is strongly elliptic.)

2. Find a holomorphic structure for S^2 (hint: use stereographic projection.)

3. Complex projective space $\mathbf{CP}^n$ is defined to be the quotient

$$\mathbf{C}^{n+1} - \{0\}/\sim,$$

where we say $(Z_0, \ldots, Z_n) \sim (Z_0', \ldots, Z_n')$ if and only if there exists $\lambda \in \mathbf{C} - \{0\}$ with $Z_i = \lambda Z_i'$ for $i = 0, \ldots, n$. Write $[Z_0, \ldots, Z_n]$ for the equivalence class of $(Z_0, \ldots, Z_n)$ (these are called 'homogeneous coordinates'). Show that $\mathbf{CP}^n$ is a complex manifold with coordinate charts (U_α, ϕ_α) for $\alpha = 1, \ldots, n$ given by

$$U_\alpha = \{[Z_0, \ldots, Z_n] \mid Z_\alpha \neq 0\}$$

and

$$\phi_\alpha([Z_0, \ldots, Z_n]) = \left(\frac{Z_0}{Z_\alpha}, \ldots, \frac{Z_{\alpha-1}}{Z_\alpha}, \frac{Z_{\alpha+1}}{Z_\alpha}, \ldots, \frac{Z_n}{Z_\alpha} \right).$$

4. Show that S^2 has exactly 3 linearly independent holomorphic vector fields.

5. Let Ω^p be the space of p-forms at a point on a Riemannian manifold M. Define a map $d : \Omega^p \to \Omega^{p+1}$ by

$$d\alpha = \frac{1}{p!} \frac{\partial}{\partial x^j} \alpha_{i_1 \cdots i_p} dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p},$$

where $\alpha_{i_1 \cdots i_p}$ is skew symmetric in $i_1, \ldots, i_p$ and α is the p-form given by

$$\alpha = \frac{1}{p!} \alpha_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}.$$

   (a) Show that d is well-defined (independent of coordinate chart).

   (b) Show that $d^2 = 0$.

6. Consider $\mathbf{CP}^n$ with the charts given as in Problem 3. In each coordinate chart U_α define

$$g^\alpha_{i\bar{j}} = \frac{\partial}{\partial z^i_\alpha} \frac{\partial}{\partial \bar{z}^j_\alpha} \log(1 + \sum_{k=1}^{n} |z^k_\alpha|^2),$$

where the $z^1_\alpha, \ldots, z^n_\alpha$ denote the coordinates for the patch U_α. Show that the $g^\alpha_{i\bar{j}}$ are the components of a Kähler metric on $\mathbf{CP}^n$. This is called the Fubini-Study metric.

7. Let M be a complex manifold and let $N \subset M$ be a complex submanifold. Let ω be a Kähler form on M and let $\iota : N \to M$ be the inclusion map. Show that $\iota^*\omega$ on N is a Kähler form on N.

8. We define the curvature tensor of a Kähler metric $g_{i\bar{j}}$ by

$$R_{p\bar{q}i\bar{j}} = -\partial_i \partial_{\bar{j}} g_{p\bar{q}} + g^{\bar{l}k} \partial_i g_{p\bar{l}} \partial_{\bar{j}} g_{k\bar{q}}.$$

Show that

$$\begin{aligned} \left[\nabla_i, \nabla_{\bar{j}}\right] \psi_r &= -g^{\bar{q}p} R_{r\bar{q}i\bar{j}} \psi_p \\ \left[\nabla_i, \nabla_{\bar{j}}\right] \psi^{\bar{s}} &= -g^{\bar{s}p} R_{p\bar{q}i\bar{j}} \psi^{\bar{q}} \\ \left[\nabla_i, \nabla_{\bar{j}}\right] \psi_{\bar{s}} &= g^{\bar{q}p} R_{p\bar{s}i\bar{j}} \psi_{\bar{q}}. \end{aligned}$$

9. Show that the Fubini-Study metric of Problem 6 is a Kähler-Einstein metric. That is, the Ricci curvature of the metric satisfies

$$R_{i\bar{j}} = \lambda g_{i\bar{j}},$$

for some $\lambda \in \mathbf{R}$. You must find λ.