Differential Geometry 230ar

Homework 5

Due: Monday December 13

Please attempt all of the problems. But you only need to hand in five of them, and you'll receive full credit if you answer all five correctly.

1. Recall that the line bundle H^l (for $l \in \mathbf{Z}$) is the line bundle on $\mathbf{CP}^n$ given by the transition functions

$$t_{\alpha\beta} = \left(\frac{Z_\beta}{Z_\alpha}\right)^l, \qquad \text{on } U_\alpha \cap U_\beta$$

(see Homework 4, problem 3 for notation). When $l = 1$ we call $H^1 = H$ the *hyperplane bundle*. Recall that a line bundle L is positive if the cohomology class $c_1(L)$ is positive (i.e. can be represented by a real positive (1,1) form). Show that $H > 0$ and $H^{-1} < 0$.

2. Let f be a homogeneous polynomial of degree d in the $n+2$ variables $Z_0, \ldots, Z_{n+1}$. Assume that f satisfies the condition

$$\cap_{i=0}^{n+1}\{Z \in \mathbf{C}^{n+2} \mid \frac{\partial f}{\partial Z_i}(Z) = 0\} = \{0, 0, \ldots, 0\}.$$

Let M_f be the n-dimensional complex submanifold of $\mathbf{CP}^{n+1}$ given by

$$M_f = \{z \in \mathbf{CP}^{n+1} \mid f(z) = 0\}.$$

Show that

$$c_1(M_f) = \frac{(n+2-d)}{\pi}[\omega_{FS}],$$

where ω_{FS} is the Kähler form of the Fubini-Study metric as described in Homework 4, problem 6.

3. Let $g_{i\bar{j}}$ be a Kähler metric on a complex manifold M with complex coordinates $z^i = x^i + \sqrt{-1}\, x^{n+i}$. Let $\omega = \frac{\sqrt{-1}}{2} g_{i\bar{j}} dz^i \wedge dz^{\bar{j}}$ be the Kähler form of g. Show that

$$\frac{\omega^n}{n!} = \det(g_{i\bar{j}})\, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^{2n}.$$

4. Stokes' Theorem on a compact manifold of (real) dimension $2n$ states that if ψ is a $(2n-1)$-form then

$$\int_M d\psi = 0.$$

Use this to prove the 'Divergence Theorem' on a compact Kähler manifold (M, ω): namely, for any smooth vector field of the type $X = X^i \frac{\partial}{\partial z^i}$, show that

$$\int_M \nabla_i X^i \frac{\omega^n}{n!} = 0.$$

5. Show that we can always find a local normal coordinate system for a Kähler metric $g_{i\bar{j}}$. That is, at any point p, there exists a holomorphic coordinate system $\{w^i\}$ with $w^i(p) = 0$ so that $g_{i\bar{j}}(0) = \delta_{ij}$ and $\frac{\partial g_{i\bar{j}}}{\partial w^k}(0) = 0$.

   (Hint: as in the real case, you can consider the change of coordinates $z^k = w^k - \frac{1}{2}\Gamma^k_{pq}(0) w^p w^q$.)

6. Recall that if $g_{i\bar{j}}$ is a Kähler metric, we can define a Riemannian metric g_R by

$$g_R = g_{i\bar{j}} dz^i \otimes d\bar{z}^j + g_{j\bar{i}} d\bar{z}^i \otimes dz^j,$$

   using $dz^i = dx^i + \sqrt{-1}\, dx^{n+i}$ and $d\bar{z}^i = dx^i - \sqrt{-1}\, dx^{n+i}$. Show

   (a) $\sqrt{\det (g_R)_{kl}} = 2^n \det (g_{i\bar{j}})$

   (b) $\sum_{k,l=1}^{2n} g_R^{kl} \nabla_k \nabla_l f = 2 \sum_{i,j=1}^{n} g^{\bar{j}i} \partial_i \partial_{\bar{j}} f$, for any smooth function f, where ∇ is the Levi-Civita connection associated to g_R.

7. Let ω and ω' be two Kähler metrics in the same Kähler class. Show that

$$\int_M \frac{\omega^n}{n!} = \int_M \frac{\omega'^n}{n!}.$$

8. Let M be a Riemann surface ($\dim_{\mathbf{C}} M = 1$) with a Kähler metric $g_{i\bar{j}}$. Consider the unnormalized Kähler-Ricci flow

$$\frac{\partial}{\partial t} g'_{i\bar{j}} = -R'_{i\bar{j}},$$

where the Kähler metric $g'_{i\bar{j}}$ is varying in time with $g' = g$ at $t = 0$. As usual, we write $\omega' = \frac{\sqrt{-1}}{2} g'_{i\bar{j}} dz^i \wedge d\bar{z}^j$. Define the volume $V = V(t)$ of the evolving metric by

$$V(t) = \int_M \omega'.$$

You may assume that the unnormalized Kähler-Ricci flow exists as long as $V(t)$ is positive (this is true, by the way). The behavior of the flow depends on the genus (the number of holes) of the underlying topological space of M:

(a) Suppose M has genus 0 (a sphere). Show that $V(t)$ shrinks to 0 in a finite time. What is that time (in terms of the initial volume)?

(b) Suppose M has genus 1 (a torus). What happens to the volume? What if the genus is greater than 1?

You will need to use the Gauss-Bonnet Theorem, which states that

$$\int_M R\,\omega = 2\pi(2 - 2g(M)),$$

where ω is any Kähler form on M, R is the associated scalar curvature, and $g(M)$ is the genus of M.