We say that a set $C \in \mathbb{R}^2$ is a smooth curve if for any $(a,b) \in C$ our set C is a smooth curve at (a,b). Analogously we say that a set $\Sigma \in \mathbb{R}^3$ is a smooth surface if for any $(a,b,c) \in \Sigma$ the set Σ is a smooth surface at (a,b,c).

1.Let $F(x,y) = x^3 + y^3 - 3xy$. For any $r \in \mathbb{R}$ we define $C_r = \{(x,y) \in \mathbb{R}^2 | F(x,y) = r\}$

a)Find for which r the level curve C_r is not smooth.

b)For any r such that the level curve C_r is not smooth find all points $(a,b) \in C_r$ such that the curve C_r is singular [not smooth] at the point (a,b).

c)Find all points $(a,b) \in \mathbb{C}^2$ such that the curve $F(x,y) = F(a,b)$ is singular [not smooth] at the point (a,b).

d)Draw the curve $C_r \subset \mathbb{R}^2$ for $r = 0$ and the case when r is a small positive and negative number.

e)Show that for any $t \in \mathbb{R}, t \neq 0, -1$ there exist unique point $x(t), y(t) \in C_0 \setminus \{(0,0)\}$ such that $y(t) = tx(t)$.

f)Show that the map $t \to \phi(t) = (x(-1+1/t), y(-1+1/t)), t \in \mathbb{R} - \{0,1\}$ extends to a parameterization $\phi : \mathbb{R} \to C_0$ of the curve C_0

g) Show that for small $\epsilon > 0$ the intersection $C_0 \cap U_\epsilon$is a union of $C' \cup C''$ where C', C'' are smooth curves which intersect the point point $(0,0)$ where $U_\epsilon = (-\epsilon, \epsilon) \times (-\epsilon, \epsilon)$

h)Find the tangent lines to C' and C'' at $(0,0)$.

2.Let $\Sigma = \{(x,y,z) \in \mathbb{R}^3 | x^2 + y^2 - z^2 = 1\}$

a)Prove that Σ is a smooth surface. Are you familiar with this surface ?

b) For any $r \in \mathbb{R}$ define $C_r = \{(x,y,z) \in \Sigma | x + y + z = r\}$. In other words C_r is the intersection of Σ with the plane $x + y + z = r$.

Find for which $r \in \mathbb{R}$ the curve C_r is not smooth. Describe the curve C_r for such $r \in \mathbb{R}$ and find singular points on C_r.

c)Do the same problem for the surface $\Sigma' = \{(x,y,z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$

3.Let $\Sigma \subset \mathbb{R}^3$ be a smooth surface such that ,$(0,0,0) \in \Sigma, f : \mathbb{R}^3 \to \mathbb{R}$ be a linear function, $C_f = \{(x,y,z) \in \Sigma | f(x,y,z) = 0\}$.

a)Prove that C_f is a smooth curve at $(0,0,0)$ if and only if the restriction f_Σ of f to the tangent plane $T_\Sigma(0,0,0)$ is not equal to 0.

b)Show that in this case the tangent line to C_f at $(0,0,0)$ is equal to $Ker f_\Sigma$.

c)EXTRA CREDIT Formulate a statement which generalizes this result.