MATH 23A
SOLUTION SET 10

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1: (a) See part (b) below.

(b) $DF(x,y) = \begin{bmatrix} 3x^2 - 3y & 3y^2 - 3x \end{bmatrix}$. Thus $DF(x,y) = 0 \implies x^2 = y$ and $y^2 = x \implies y = x^2 = (y^2)^2 = y^4 \implies y = 0$ or 1 (since $y \in \mathbb{R}$) $\implies (x,y) = (0,0)$ or $(1,1)$. One can check that these pairs do make $DF(x,y) = 0$. Therefore, the set of real critical points is $\{(0,0),(1,1)\}$. This set consists of isolated points, so the complex level curves containing $(0,0)$ or $(1,1)$ are singular at these points. For the purposes of this course, you can say that this implies that the *real* level curves are singular, though this conclusion doesn't really follow.

The answer: $C_{F(0,0)} = C_0$ is smooth except at $(0,0)$, $C_{F(1,1)} = C_{-1}$ is smooth except at $(1,1)$, and C_r is smooth for any $r \in \mathbb{R} - \{0,-1\}$.

The later parts of the problem will show that C_0 really is singular at $(0,0)$. As for C_{-1}, one can show that $(1,1)$ is an isolated point in the curve. (All that follows is *not* required for your solution.) Say $\epsilon > 0$ is small. We claim that the equation $F(1+\epsilon, y) = -1$ has no solution near -1. The equation is equivalent to $y^3 - 3(1+\epsilon)y + (1+\epsilon)^3 + 1 = 0$. Let $f_1(y) = y^3 - 3(1+\epsilon)y + (1+\epsilon)^3 + 1$. Then $f_1'(y) = 3y^2 - 3(1+\epsilon)$, so around 1, $f_1(y)$ has a minimum value at $y = \sqrt{1+\epsilon}$. But this yields $f_1(\sqrt{1+\epsilon}) = -2(1+\epsilon)^{\frac{3}{2}} + (1+\epsilon)^3 + 1 = \left((1+\epsilon)^{\frac{3}{2}} - 1\right)^2 > 0$. Hence f_1 is positive around $y = 1$, and C_{-1} has no point of the form $(1+\epsilon, y)$ near $(1,1)$. As for $1-\epsilon$, one can show that near 1, the function $f_2(y) = F(1-\epsilon, y) + 1$ achieves its minimum at $\sqrt{1-\epsilon}$, and that $f_2(\sqrt{1-\epsilon}) = \left((1-\epsilon)^{\frac{3}{2}} - 1\right)^2 > 0$. Lastly, we note that the same argument will work for $(x, 1+\epsilon)$ and $(x, 1-\epsilon)$ by symmetry. Therefore, C_{-1} does indeed have an isolated point at $(1,1)$.

(c) $DF(x,y)$ is still $\begin{bmatrix} 3x^2 - 3y & 3y^2 - 3x \end{bmatrix}$. Thus $DF(x,y) = 0 \implies x^2 = y$ and $y^2 = x \implies y = x^2 = (y^2)^2 = y^4$. However, y is now allowed to be complex, so y can be 0, 1, $\frac{-1+\sqrt{3}i}{2}$, or $\frac{-1-\sqrt{3}i}{2}$. The relation $x = y^2$ implies that (x,y) must be $(0,0)$, $(1,1)$, $(\frac{-1+\sqrt{3}i}{2}, \frac{-1-\sqrt{3}i}{2})$, or $(\frac{-1-\sqrt{3}i}{2}, \frac{-1+\sqrt{3}i}{2})$. One can check that all four of these pairs satisfy $DF(x,y) = 0$. Hence this is the set of critical points. Because it consists of isolated points, by a theorem given in class the level curves containing these points are singular at these points.

(d) See the attached Mathematica worksheet.

(e) The problem is equivalent to showing that given $t \in \mathbb{R} - \{0,-1\}$, there exists unique $(x,y) \in \mathbb{R}^2$ such that $F(x,y) = 0$, $(x,y) \neq (0,0)$, and $y = tx$. This amounts to solving two equations. If (x,y) is a solution, then $0 = x^3 + y^3 - 3xy = x^3 + (tx)^3 - 3x(tx) = (t^3+1)x^3 - 3tx^2 \implies x = 0$ or $\frac{3t}{t^3+1}$ (the latter well-defined because $t \neq -1$). But if x were 0, then $y = tx = 0$, violating the condition that $(x,y) \neq (0,0)$. Therefore, $x = \frac{3t}{t^3+1}$ and $y = tx = \frac{3t^2}{t^3+1}$. This is the only possible solution, and it's easy to check that it works. (Note that it's nonzero because $t \neq 0$.)

Note that because such (x, y) is unique for each $t \in \mathbb{R} - \{0, -1\}$, we can denote as $(x(t), y(t))$ the (x, y) obtained for a particular t. Hence $x(t) = \frac{3t}{t^3+1}$ and $y(t) = \frac{3t^2}{t^3+1}$. These will be used in the later parts.

(f) First we show that as t ranges in $\mathbb{R} - \{0, -1\}$, $(x(t), y(t))$ covers all of $C_0 - \{(0, 0)\}$. Suppose $(a, b) \in C_0 - \{(0, 0)\}$. Then a and b are nonzero (because $a = 0 \implies 0 = F(a, b) = b^3 \implies b = 0$; similar for $b = 0$). Further, $b/a \neq -1$ (because $b = -a \implies 0 = F(a, b) = 3a^2 \implies a = b = 0$). Hence (a, b) satisfies the conditions $F(a, b) = 0$, $(a, b) \neq (0, 0)$, and $b = (b/a)a$ for $b/a \in \mathbb{R} - \{0, -1\}$. By part (e), (a, b) is the unique point that satisfies all these conditions for $t = b/a$. Hence $(a, b) = (x(b/a), y(b/a))$. But as t ranges in $\mathbb{R} - \{0, 1\}$, $-1 + 1/t$ ranges in $\mathbb{R} - \{0, -1\}$ (check this). Therefore, $(x(-1 + 1/t), y(-1 + 1/t))$ ranges in $C_0 - \{(0, 0)\}$, as claimed.

Now

$$\begin{aligned}
\varphi(t) &= \left(\frac{3(-1 + 1/t)}{(-1 + 1/t)^3 + 1}, \frac{3(-1 + 1/t)^2}{(-1 + 1/t)^3 + 1} \right) \\
&= \left(\frac{-3t^3 + 3t^2}{(-t + 1)^3 + t^3}, \frac{3t(-t + 1)^2}{(-t + 1)^3 + t^3} \right) \qquad \text{(multiplying top and bottom by } t^3\text{)} \\
&= \left(\frac{3t^2(1 - t)}{3t^2 - 3t + 1}, \frac{3t(1 - t)^2}{3t^2 - 3t + 1} \right).
\end{aligned}$$

Since $3t^2 - 3t + 1$ is always positive (why?), the above fractions' denominators are nonzero for any t. Therefore, we can *define* a function $\bar{\varphi} \colon \mathbb{R} \to \mathbb{R}^2$ by letting $\varphi(t) = \left(\frac{3t^2(1-t)}{3t^2-3t+1}, \frac{3t(1-t)^2}{3t^2-3t+1} \right)$. This way, $\bar{\varphi}$ is certainly continuously differentiable (being a quotient of two continuously differentiable functions where the denominator is never zero), *plus* it has the property of being an extension of the original $\varphi \colon \mathbb{R} - \{0, 1\} \to \mathbb{R}^2$.

The original φ covers all of $C_0 - \{(0, 0)\}$ as noted above, so $\bar{\varphi}$ does as well. But in addition, $\bar{\varphi}(0) = (0, 0) = \bar{\varphi}(1)$. Hence the image of $\bar{\varphi}$ is exactly C_0, and $\bar{\varphi}$ is a parametrization for C_0.

Note: It is a common technique to define something first, then to prove it has the desired properties. This is often useful for proving existence.

(g) We've shown in (f) that $\bar{\varphi}$ parametrizes C_0. We claim that it is one-to-one except for $\bar{\varphi}(0) = \bar{\varphi}(1) = (0, 0)$. The key is to show that $\varphi \colon \mathbb{R} - \{0, 1\} \to \mathbb{R}^2$ is one-to-one: the function $t \mapsto -1 + 1/t$ is one-to-one, and the function $u \mapsto (x(u), y(u))$ is one-to-one as well (since $(x(u), y(u)) = (x(u'), y(u')) \implies u = y(u)/x(u) = y(u')/x(u') = u'$). Therefore, the composition $t \mapsto \varphi(t) = (x(-1 + 1/t), y(-1 + 1/t))$ is also one-to-one. $\bar{\varphi}$ is just φ defined at two extra values of t, so our claim is proved.

Next we claim that $D\bar{\varphi}$ is nonzero near 0 and near 1. Calculations show that $D\bar{\varphi}(t) = \left[\frac{3t(-3t^2+6t^2-6t+2)}{(3t^2-3t+1)^2} \quad \frac{3(1-t)(3t^3-3t^2+3t-1)}{(3t^2-3t+1)^2} \right]$. So in particular $D\bar{\varphi}(0)$ and $D\bar{\varphi}(1)$ are nonzero. By the continuity of $D\bar{\varphi}$, it is nonzero near 0 and 1. Our second claim is proved.

Now we know that the image of $\bar{\varphi}$ is C_0. Let $\epsilon > 0$ be small. Let $\bar{x}(t) = \frac{3t^2(1-t)}{3t^2-3t+1}$ and $\bar{y}(t) = \frac{3t(1-t)^2}{3t^2-3t+1}$. By one-variable calculus techniques, one can show that $\{t : \bar{x}(t) \in (-\epsilon, \epsilon)\}$ is of the form $(-\delta_1, \delta_2) \cup (1-\delta_3, 1+\delta_4)$ for some small positive $\delta_1, \delta_2, \delta_3, \delta_4$, and that $\{t : \bar{y}(t) \in (-\epsilon, \epsilon)\}$ is of the form $(-\delta_5, \delta_6) \cup (1-\delta_7, 1+\delta_8)$. Hence

$$\begin{aligned}\{t : \bar{\varphi}(t) = (\bar{x}(t), \bar{y}(t)) \in U_\epsilon\} &= ((-\delta_1, \delta_2) \cup (1-\delta_3, 1+\delta_4)) \cap ((-\delta_5, \delta_6) \cup (1-\delta_7, 1+\delta_8)) \\ &= (-\delta_9, \delta_{10}) \cup (1-\delta_{11}, 1+\delta_{12})\end{aligned}$$

for some small positive $\delta_9, \ldots, \delta_{12}$. Let $I_1 = (-\delta_9, \delta_{10})$ and $I_2 = (1-\delta_{11}, 1+\delta_{12})$. Recall that $D\bar{\varphi}$ is nonzero close to 0 and 1. We can force I_1 and I_2 to be sufficiently close to 0 and 1 by making the original ϵ sufficiently small. Hence there exists $\epsilon > 0$ such that $D\bar{\varphi}$ would be nonzero in I_1 and in I_2; we pick such an ϵ at the very beginning.

Therefore, $C_0 \cap U_\epsilon = \bar{\varphi}(I_1 \cup I_2) = \bar{\varphi}(I_1) \cup \bar{\varphi}(I_2)$. But $\bar{\varphi}|_{I_1}$ and $\bar{\varphi}|_{I_2}$ are one-to-one, and $D\bar{\varphi} \neq 0$ in I_1 and I_2. Hence by the parameterization definition of a smooth curve, the image C' of $\bar{\varphi}|_{I_1}$ and the image C'' of $\bar{\varphi}|_{I_2}$ are smooth curves. The above discussion shows that $C_0 \cap U_\epsilon = C' \cup C''$ and that $C' \cap C'' = \{(0,0)\}$.

(h) The tangent line to C' at $(0,0)$ (corresponding to $t = 0$) is given by the parameterization $(D_1\bar{\varphi}(0)t + 0, D_2\bar{\varphi}(0)t + 0) = (0t + 0, -3t + 0) = (0, -3t)$. This corresponds to the line $x = 0$.

The tangent line to C'' at $(0,0)$ (corresponding to $t = 1$) is given by the parameterization $(D_1\bar{\varphi}(1)t + 0, D_2\bar{\varphi}(1)t + 0) = (-3t + 0, 0t + 0) = (-3t, 0)$. This corresponds to the line $y = 0$.

2: (a) Let $f = x^2 + y^2 - z^2$. Then Σ is a level set of f. Now $Df(x,y,z) = \begin{bmatrix} 2x & 2y & -2z \end{bmatrix}$, which is zero iff $x = y = z = 0$. However, $(0,0,0)$ is not in Σ. Hence Df is nonzero in Σ, implying that Σ is a smooth surface by the level set definition of a smooth surface in $\mathbb{R}^3$.

The surface looks like an hourglass (an apt description provided by some of you).

(b) Define $F\colon \mathbb{R}^3 \to \mathbb{R}^2$ by $F(x,y,z) = \begin{bmatrix} f(x,y,z) \\ x+y+z \end{bmatrix} = \begin{bmatrix} x^2+y^2-z^2 \\ x+y+z \end{bmatrix}$. Then C_r is a level set of F: $C_r = \left\{ (x,y,z) : F(x,y,z) = \begin{bmatrix} 1 \\ r \end{bmatrix} \right\}$. Hence if for a given r, $DF(x,y,z)$ is onto for any $(x,y,z) \in C_r$, then C_r is a smooth curve by the level set definition of a smooth curve in $\mathbb{R}^3$. Now $DF(x,y,z) = \begin{bmatrix} 2x & 2y & -2z \\ 1 & 1 & 1 \end{bmatrix}$. If any two of $2x$, $2y$, and $-2z$ are nonequal, then the two corresponding rows are linearly independent (why?) and $DF(x,y,z)$ is onto. The only time this doesn't happen is when $2x = 2y = -2z \implies 1 = f(x,y,z) = x^2 + x^2 - (-x)^2 = x^2 \implies x = 1$ or $-1 \implies r = x + x + (-x) = 1$ or -1. Hence for any r not equal to 1 or -1, $DF(x,y,z)$ is onto in all of C_r, and C_r is a smooth curve.

But we have yet to settle the cases $r = 1$ or -1. First suppose $r = 1$. Then $x+y+z = 1$ and $x^2+y^2-z^2 = 1 \implies 1 = x^2 + y^2 - (1-x-y)^2 = -2xy + 2x + 2y - 1 \implies 0 = xy - x - y + 1 = (x-1)(y-1) \implies x = 1$ or $y = 1$. But once $x = 1$, the two equations reduce to $y+z=0$ and $y^2-z^2=0$, so that y can be anything and z must be $-y$. Similarly, if $y=1$, then x can be anything and z must be $-x$. Thus $C_1=\{(1,y,-y):y\in\mathbb{R}\}\cup\{(x,1,-x):x\in\mathbb{R}\}$. This is just a pair of lines in $\mathbb{R}^3$ that intersect at $(1,1,-1)$. Hence C_1 is not smooth and has a sole singular point $(1,1,-1)$.

Lastly suppose $r=-1$. Then $x+y+z=-1$ and $x^2+y^2-z^2=1 \implies 1=x^2+y^2-(-1-x-y)^2=-2xy-2x-2y-1 \implies 0=xy+x+y+1=(x+1)(y+1) \implies x=-1$ or $y=-1$. From this we can deduce that $C_{-1}=\{(-1,y,-y):y\in\mathbb{R}\}\cup\{(x,-1,-x):x\in\mathbb{R}\}$. This is just a pair of lines in $\mathbb{R}^3$ that intersect at $(-1,-1,1)$. Hence C_{-1} is not smooth and has a sole singular point $(-1,-1,1)$.

(c) Let $g=x^2+y^2-z^2$. Then Σ' is a level set of g. Now $Dg(x,y,z)=\begin{bmatrix}2x & 2y & 2z\end{bmatrix}$, which is zero iff $x=y=z=0$. However, $(0,0,0)$ is not in Σ'. Hence Dg is nonzero in Σ', implying that Σ' is a smooth surface by the level set definition of a smooth surface in $\mathbb{R}^3$.

The surface looks like a hollow sphere, of radius 1 around the origin.

Now define $G\colon\mathbb{R}^3\to\mathbb{R}^2$ by $G(x,y,z)=\begin{bmatrix}g(x,y,z)\\ x+y+z\end{bmatrix}=\begin{bmatrix}x^2+y^2+z^2\\ x+y+z\end{bmatrix}$. Then $C'_r=\{(x,y,z)\in\Sigma':x+y+z=r\}$ is a level set of G: $C'_r=\left\{(x,y,z):G(x,y,z)=\begin{bmatrix}1\\ r\end{bmatrix}\right\}$. Hence if for a given r, $DG(x,y,z)$ is onto for any $(x,y,z)\in C'_r$, then C'_r is a smooth curve by the level set definition of a smooth curve in $\mathbb{R}^3$. Now $DG(x,y,z)=\begin{bmatrix}2x & 2y & 2z\\ 1 & 1 & 1\end{bmatrix}$. If any two of $2x$, $2y$, and $2z$ are nonequal, then the two corresponding rows are linearly independent (why?) and $DF(x,y,z)$ is onto. The only time this doesn't happen is when $2x=2y=2z \implies 1=f(x,y,z)=x^2+x^2+x^2=3x^2 \implies x=\frac{\sqrt{3}}{3}$ or $-\frac{\sqrt{3}}{3} \implies r=x+x+x=\sqrt{3}$ or $-\sqrt{3}$. Hence for any r not equal to $\sqrt{3}$ or $-\sqrt{3}$, $DG(x,y,z)$ is onto in all of C'_r, and C'_r is a smooth curve.

Now we take care of the cases $r=\sqrt{3}$ or $-\sqrt{3}$. First suppose $r=\sqrt{3}$. Then (x,y,z) is in C'_r iff it satisfies $x+y+z=\sqrt{3}$ and $x^2+y^2+z^2=1$. However, the only solution is $x=y=z=\frac{\sqrt{3}}{3}$. (One way to see this is to note that $0=1-2+1=(x^2+y^2+z^2)-\frac{2\sqrt{3}}{3}(x+y+z)+1=(x-\frac{\sqrt{3}}{3})^2+(y-\frac{\sqrt{3}}{3})^2+(z-\frac{\sqrt{3}}{3})^2$, so that each square must be 0.) Hence $C'_{\sqrt{3}}=\left\{(\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3})\right\}$. It is now clear that $C'_{\sqrt{3}}$ is not smooth and has the sole singularity point $(\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3})$.

The case $r=-\sqrt{3}$ is similar and yields $C'_{-\sqrt{3}}=\left\{(-\frac{\sqrt{3}}{3},-\frac{\sqrt{3}}{3},-\frac{\sqrt{3}}{3})\right\}$. Thus $C'_{-\sqrt{3}}$ is not smooth and has the sole singularity point $(-\frac{\sqrt{3}}{3},-\frac{\sqrt{3}}{3},-\frac{\sqrt{3}}{3})$.

3: (a) One direction of this problem is actually false: C_f being smooth at $(0,0,0)$ does not imply that $f_\Sigma\neq 0$. Here is an example courtesy of Nils: Let $g(x,y,z)=x^2+(y-1)^2$ and let $\Sigma=\{(x,y,z):g(x,y,z)=1\}$. Let $f(x,y,z)=x$. Then C_f is the intersection of Σ with $\ker f$ (the yz plane); it turns out to be the union of the lines $\{(0,0,z):z\in\mathbb{R}\}$ and $\{(0,2,z):z\in\mathbb{R}\}$, hence a smooth curve. However, the tangent plane $T_\Sigma(0,0,0)$ is $y=0$, on which f is *not* identically 0.

The other direction is true. Because Σ is a smooth surface, there exists a continuously differentiable function $g\colon\mathbb{R}^3\to\mathbb{R}$ such that

- $Dg(0,0,0) \neq 0$ and
- near $(0,0,0)$, Σ coincides with the level set $\{(x,y,z) : g(x,y,z) = 0\}$.

Define $F\colon \mathbb{R}^3 \to \mathbb{R}^2$ by $F(x,y,z) = \begin{bmatrix} g(x,y,z) \\ f(x,y,z) \end{bmatrix}$. Then near $(0,0,0)$, C_f coincides with the level set $\left\{(x,y,z) : F(x,y,z) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$. Hence to show that C_f is smooth at $(0,0,0)$, it suffices to show that $DF(0,0,0)$ is onto.

Let $f(x,y,z) = ax + by + cz$. Then

$$DF(0,0,0) = \begin{bmatrix} D_1g(0,0,0) & D_2g(0,0,0) & D_3g(0,0,0) \\ a & b & c \end{bmatrix}.$$

Thus $DF(0,0,0)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} D_1g(0,0,0)x + D_2g(0,0,0)y + D_3g(0,0,0)z \\ ax + by + cz \end{bmatrix}$ $= \begin{bmatrix} D_1g(0,0,0)x + D_2g(0,0,0)y + D_3g(0,0,0)z \\ f(x,y,z) \end{bmatrix}$. Now the tangent plane $T_\Sigma(0,0,0)$ is exactly the set $\{(x,y,z) : D_1g(0,0,0)x + D_2g(0,0,0)y + D_3g(0,0,0)z = 0\}$, so because f is not identically 0 on $T_\Sigma(0,0,0)$, there exists (x_1,y_1,z_1) such that

- $D_1g(0,0,0)x_1 + D_2g(0,0,0)y_1 + D_3g(0,0,0)z_1 = 0$ and
- $f(x_1,y_1,z_1) = r \neq 0$.

Hence $DF(0,0,0)\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix}$.

Now, because $Dg(0,0,0) \neq 0$, there exists (x_2,y_2,z_2) such that $D_1g(0,0,0)x_2 + D_2g(0,0,0)y_2 + D_3g(0,0,0)z_2 = s \neq 0$. Thus $\begin{bmatrix} s \\ f(x_2,y_2,z_2) \end{bmatrix} = DF(0,0,0)\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ r \end{bmatrix} = DF(0,0,0)\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ are in the image of $DF(0,0,0)$. But these two vectors form a basis for $\mathbb{R}^2$ because $s \neq 0$ and $t \neq 0$. Therefore, the image of $DF(0,0,0)$ is all of $\mathbb{R}^2$, i.e. $DF(0,0,0)$ is onto. As discussed above, this implies that C_f is smooth at $(0,0,0)$.

(b) As discussed in lecture, the tangent line to C_f at $(0,0,0)$ is

$$\begin{aligned}
\ker DF(0,0,0) &= \{(x,y,z) : D_1g(0,0,0)x + D_2g(0,0,0)y + D_3g(0,0,0)z \text{ and } f(x,y,z) = 0\} \\
&= \{(x,y,z) : (x,y,z) \in T_\Sigma(0,0,0) \text{ and } f(x,y,z) = 0\} \\
&= T_\Sigma(0,0,0) \cap \ker f \\
&= \ker f_\Sigma.
\end{aligned}$$