MATH 23B PROBLEM SET 1

due February 11th

Let S_n be the set of permutations of the set $[1, ..., n]$. We denote by e the *trivial* permutation $e(i) \equiv i$. Given two permutations $\sigma', \sigma'' \in S_n$ we define a permutation $\sigma := \sigma' \circ \sigma'' \in S_n$ by

$$\sigma(i) = \sigma'(\sigma''(i)), 1 \leq i \leq n$$

. For any $i, 1 \leq i \leq n-1$ we denote by $s_i \in S_n$ a permutation which interchanges the elements i and $i+1$ and fixes all other elemnts $j, 1 \leq j \leq n$. We call the elements $s_i \in S_n$ *simple reflections.*

1.a) Prove that any element $\sigma \in S_3$ can be written as a composition of elements $s_1, s_2 \in S_3$.

b) Prove that any element $\sigma \in S_n$ can be written as a composition of simple reflections $s_1, ..., s_{n-1} \in S_n$.

For any $\sigma \in S_n$ we denote by $l(\sigma)$ the number of pairs $1 \leq i < j \leq n$ such that $\sigma(i) > \sigma(j)$.

2.a) Show that for any $\sigma \in S_n$ and $i, 1 \leq i \leq n-1$ either we have $l(s_i \circ \sigma) = l(\sigma) + 1$ or $l(s_i \circ \sigma) = l(\sigma) - 1$.

b)Show the existence of a map $sign : S_n \to \pm 1$ such that $sign(e) = 1$ and for any $\sigma \in S_n$ and $i, 1 \leq i \leq n-1$ we have $sign(s_i \circ \sigma) = -sign(\sigma)$.

c)Show that such a map $sign : S_n \to \pm 1$ is unique.

d)Show that for any $\sigma', \sigma'' \in S_n$ we have $sign(\sigma' \circ \sigma'') = sign(\sigma')sign(\sigma'')$.

3. Show that any $\sigma \in S_n$ can be written as a composition of $l(\sigma)$ simple reflections and can not be written as a product of a smaller number of simple reflections.