MATH 23A 1ST MIDTERM EXAM, DUE OCT. 20

1.Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

a)Find A^7.

b) Show that for any real polynomial $P(t) = \sum_{i=0}^{n} p_i t^i$ we have $P(A) = p(0)Id + p'(0)(A - Id)$ where $P(A) := \sum_{i=0}^{n} p_i A^i$.

c)Show that any 2×2 matrix B such that $AB = BA$ has a form $B = P(A)$ for some polynomial $P(t) = \sum_{i=0}^{n} p_i t^i$.

d)EXTRA CREDIT. Generalize the parts b) and c) to the case when A is a 3×3 matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

2.Let A, B be two 2×2 matricies such that $AB = 0$. Does this equlity imply that $BA = 0$?

3. Let $V := \{(x_1, ..., x_n) \in \mathbb{R}^n \,|\, \sum_{i=1}^{n} x_i = 0\} \subset \mathbb{R}^n$. Prove that V is a subspace of $\mathbb{R}^n$ and find a basis for V.

We say that an $n \times n$-matrix $A = (a_{i,j}), 1 \leq i, j \leq n$ is symmetric if $a_{i,j} = a_{j,i}, 1 \leq i, j \leq n$ and we say that A is skew-symmetric if $a_{i,j} = -a_{j,i}, 1 \leq i, j \leq n$. We denote by $Sym^2(\mathbb{R}^n) \subset \mathcal{M}(n.n)$ and $\Lambda^2(\mathbb{R}^n) \subset \mathcal{M}(n.n)$ the subsets of symmetric and skew-symmetric matricies.

4. Prove that the subsets $Sym^2(\mathbb{R}^n) \subset \mathcal{M}(n.n)$ and $\Lambda^2(\mathbb{R}^n) \subset \mathcal{M}(n.n)$ are linear subspaces of of and find the dimensions of these subspaces.

5. Let $A = (a_{i,j}), 1 \leq i, j \leq 3$ be a 3×3 matric such that $a_{i,j} \equiv 1$. Consider the set $V = \{B \in \mathcal{M}(3.3) | AB = \bar{0}\}$.

Prove that V is a subspace of $\mathcal{M}(3.3)$ and find the dimension of V.

6. For any basis $B = \{(\bar{v}_1, ..., \bar{v}_n)\})$ in a vector space V we denote by $T_B : \mathbb{R}^n \to V$ an isomorphism of vector spaces such that $T(\bar{e}_i) = \bar{v}_i, 1 \leq i \leq n$. Let $B' = \{(\bar{v}'_1, ..., \bar{v}'_n)\})$ be another basis of V. Then we can find an $n \times n$-matrix $A = (a_{i,j}), 1 \leq i, j \leq n$ such that $\bar{v}_j = \sum_{i=1}^{n} a_{i,j} \bar{v}'_i$. Let $T := T_{B'}^{-1} T_B : \mathbb{R}^n \to \mathbb{R}^n$. Show that $A = A_T$ where A_T is the $n \times n$-matrix corresponding to T as in section 2.6