HOMEWORK FOR DEC. 20-th Let V be a finite-dimensional vector space and q a real-valued function on V. We say that q is quadratic if for any two vectors $v, w \in V$ there exists numbers $\alpha_k(v,w), 0 \le k \le 2$ such that

$$q(av+bw) = \sum_{k=0}^{2} \alpha_k(v,w) a^k b^{2-k}$$

Let $B = \{e_1, ..., e_d\}$ be a basis of V 1. a)Show q is a quadratic polynomial if and only if there exists numbers $c^B_{i,j} = c^B_{i,j}(f), 1 \le i \le j \le d \in \mathbb{R}$ such that for any $x^1, ..., x^d \in \mathbb{R}$ we have $f(\sum_{i=1}^d x^i e_i) = \sum_{1 \le i \le j \le d}^d c^B_{i,j} x^i x^j x_i$.

b)Let $V = P_N$ be the space of polynomial $p(t)$ of degree $\le N, B = \{e_1, ..., e_d\}$ where $e_i := \{t^i\}, 0 \le i \le N$ and $q : P_N \to \mathbb{R}$ be the function given by $q(p) := p(1)^2$. Show that q is a quadratic polynomial and find the coefficients [the numbers] $c^B_{i,j}(q), 1 \le i \le j \le d$.

c)Let $A : P_N \to P_N$ be a linear map given by $A(p)(t) = p'(t)$. Show that the function $A^\vee(q) : P_N \to \mathbb{R}, A^\vee(q)(p) := q(A(p))$ is quadratic and find the coefficients [the numbers] $c^B_{i,j}(A^\vee(q))$.

For any basis $B = \{e_1, ..., e_d\}$ of V we denote by $T_B : \mathbb{R}^d \to V$ the isomorphism such that $T_B(x^1, ..., x^d) := \sum_{i=1}^d x^i e_i$. If $F : V \to \mathbb{R}$ is a function on V we denote by $F_B : \mathbb{R}^d \to \mathbb{R}$ the function given by $F_B(\bar{x}) := F(T_B(\bar{x}))$. [That is $F_B = F \circ T_B$].

Definition. We say that a function $F : V \to \mathbb{R}$ is twice differentiable at $v \in V$ if for some basis B of V the function $F_B : \mathbb{R}^d \to \mathbb{R}$ is twice differentiable at $F_B^{-1}(v) \in \mathbb{R}^d$.

2. a) Show that if for some basis B of V the function $F_B : \mathbb{R}^d \to \mathbb{R}$ is twice differentiable at $F_B^{-1}(v) \in \mathbb{R}^d$ then for any basis B' of V the function $F_{B'} : \mathbb{R}^d \to \mathbb{R}$ is twice differentiable at $F_{B'}^{-1}(v) \in \mathbb{R}^d$.

b)Show that a function $F : V \to \mathbb{R}$ is twice differentiable at $v \in V$ iff [if and only if] there exists a quadratic function $q(v)$ on $V, q(v) : V \to \mathbb{R}$ of degree 2 such that

$$\|(F(v+h) - F(v) - dF(v)(h) - q(v)(h)\| / \|h\|^2 \to 0$$

for $\|h\| \to 0, h \in V$ where $dF(v) : V \to \mathbb{R}$ is the differential of F at v.

c)Choose a basis B of V and show how to espress the coefficients $c^B_{i,j}(q)(v)$ in term of the second partial derivatives of the function F_B.

3. Let $f \in \mathcal{B}^2_s(V)$ be a symmetric bilinear form on V. We define a function q_f on V by $q_f(v) := f(v,v), v \in V$.

a) Show that q_f is a quadratic function on V.

b)Let V and q be as in 1b). Find a symmetric bilinear form f on V such that $q = q_f$.

c)Let q be a quadratic function on V. Show that there exists unique symmetric bilinear form f on V such that $q = q_f$

A hint. Given q you can write an explicit formula for f.

4.a) Let $f : V \to \mathbb{R}$ be a function. Finish the following definition.

F is a homogenious polynomial on V of degree n if...

b)Generalize the questions of the problem to the case when f is a symmetric n-linear form on V.

c) Find answers to these questions.

Let V be a vector space of dimension d. For any $r \geq 0$ we denote by $\Omega^r(V)$ the space of antisymmetric r-linear forms on V

Remark. By the definition $\Omega^1(V) = V^\vee$ and $\Omega^2(V) = \mathcal{B}_{as}$. We define $\Omega^0(V) := \mathbb{R}$.

Let $\mathcal{I}(r, d)$ be the set of all sequences $I = (i_1, ...i_r), 1 \leq i_1 < ... < i_r \leq d$.

5. a)Find the number of elements in the set $\mathcal{I}(r, d)$.

Fix a basis $\{e_1, ..., e_d\}$ of V. For any $I = (i_1, i_2) \in \mathcal{I}(2, d)$ we define a bilinear form $\omega^I \in \Omega^2(V)$ by $\omega^I(v_1, v_2) := e^1(v_1)e^2(v_2) - e^1(v_2)e^2(v_1)$.

b)Prove that the set $\omega^I, I =\in \mathcal{I}(2, d)$ is a basis of the space $\Omega^2(V)$.

c)Construct a basis $\omega^I, I =\in \mathcal{I}(3, d)$ in the space $\Omega^2(V)$ and find the dimension of the space $\Omega^3(V)$.

d)Construct a basis $\omega^I, I =\in \mathcal{I}(r, d)$ in the space $\Omega^2(V)$ and find the dimension of the space $\Omega^r(V)$.