Homework for Nov. 20th

1. For each of the following concepts, give the definition and then give its negation, e.g., for the first part, define what it means for 2 norms to be equivalent and then state what you would have to prove to show that 2 norms are not equivalent (which is the same thing as what you would assume at the beginning of a proof by contradiction to show that two norms were equivalent).

a) Two norms, $\|v\|$ and $\|v\|'$ on a vector space V are equivalent.

b) A function $f : V \to W$ from the normed vector space $(V, \|v\|_V)$ to the normed vector space $(W, \|w\|_W)$ is continuous.

c) A subset $A \subset V$ of a normed vector space $(V, \|v\|_V)$ has the finite subcovering property.

2. Use Taylor's Theorem to prove the Binomial Theorem, which states that

$$(a+b)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} a^k b^{n-k}$$

where $a, b \in \mathbb{R}$ and n is a positive integer.

The next two problems give different ways of proving that the image of a compact set under a continuous map is compact. Each one must be done by the method indicated, and you cannot use the results of one to help you with the other (in particular, you can't just do number 6 and then point out that it implies number 5). You also cannot quote any results from Chapter 5 Section 4 of the book (although you can look at it to see a third way of proving this result). Let $(V, \|v\|_V)$ and $(W, \|w\|_W)$ be normed vector spaces, and let $f : V \to W$ be a continuous function (with respect to the given norms) and let $A \subset V$ be a compact set.

3. Assume that both V and W are finite-dimensional. Then sets in either vector space are compact if and only if they are closed and bounded. Prove that $f(A) = \{w \in W | f(v) = w \text{ for some } v \in V\}$ is closed and bounded (you will probably want to use Theorem 1.7 of Chapter 5, on page 137, which is interesting in its own right), and thus conclude that $f(A)$ is compact.

4. In this problem, we no longer assume that V and W are finite-dimensional. Nonetheless, we know that sets in either space are compact if and only if they possess the finite subcovering property. Using the fact that A has this property, along with what we know about the

behavior of open sets with regard to continuous maps, show that $f(A)$ has the finite subcovering property. Conclude that $f(A)$ is compact.

Finally, do the following three problems from the book:
6.4.3 (page 185)
6.7.4 (page 199)
6.8.2 (page 201)