Chapter 3: Section 3, Number 10

A SOLUTION.

a) Let $x \in \mathbb{R}$ be a rational number. We want to show that f is not continuous at x. So we want to show that there exists $\epsilon > 0$ such that for any $\delta > 0$ we can find $y, |y - x| < \delta$ such that $|f(x) - f(y)| > \epsilon$.

Choose $\epsilon := f(x)/2$. For any $\delta > 0$ we can find $y, |y - x| < \delta$ such that y is irrational. Then $f(y) = 0$ and $|f(x) - f(y)| = f(x) > \epsilon$.

b) Let $x \in \mathbb{R}$ be an irrational number. We want to show that f is continuous at x. So we want to show that for any $\epsilon > 0$ there exists $\delta > 0$ such that for any $y \in \mathbb{R}, |y - x| < \delta$ we have $|f(x) - f(y)| < \epsilon$.

Fix any $\epsilon > 0$ and choose an integer n such $1/n < \epsilon$. Let Z be set of all rational numbers p/q such that $q \leq n$ and $|x - p/q| < 1$. It is clear that the set Z is finite. Since x is irrational x does not belong to Z. Therefore we can find $\delta > 0$ such that $|x - z| > \delta$ for all $z \in Z$. We assume that $\delta < 1$.

Let $y \in \mathbb{R}$ be a number such that $|y - x| < \delta$. Then y does not belong to Z. So either y is irrational or $= p/q$ where $q > n$. In any case $|f(y)| < 1/n < \epsilon$. Since $f(x) = 0$ we have $|f(x) - f(y)| < \epsilon$.

Chapter 5: Section 2, Number 9

A SOLUTION.

We assume that no such r exists and show that this assumption leads to a contrudiction.

So assume that no such r exists. Then for any $r > 0$ we can find a point $v \in C$ such that for any $\lambda \in \Lambda, B_r(v) \not\subseteq U_\lambda$. In particular, for any $n > 0$ we can find $v_n \in C$ such that $B_{1/n}(v_n) \not\subseteq U_\lambda$ for any $\lambda \in \Lambda$.

Consider the sequence $v_n \in C, n > 0$. Since C is compact we can choose a subsequence $v_{n_m}, m > 0, n_{m+1} > n_m$ which is convergent to $v \in C$. We define $c_m := v_{n_m}$. Since N_m is monotonely increasing we have $n_m \geq m$. Therefore our asumptions show that

$*$ For any $m > 0$ and any $\lambda \in \Lambda$ we have $B_{1/m}(c_n) \not\subseteq U_\lambda$.

Since $U_\lambda, \lambda \in \Lambda$ is a covering of C we can find $\lambda_0 \in \Lambda$ such that $v \in U_{\lambda_0}$. Since the set U_{λ_0} is open we can find $\epsilon > 0$ such that $B_\epsilon(v) \subset U_{\lambda_0}$. Since the sequence c_m is convergent to v we can find $m_0 > 0$ such that $c_m \in B_{\epsilon(v)/2}$ for all $m > m_0$. Choose $m > max(m_0, 2/\epsilon)$. Since $B_{1/m}(c_m) \subset B_\epsilon(v) \subset U_{\lambda_0}$ we see that $B_{1/m}(c_m) \subset U_{\lambda_0}$. But this contrudicts $*$. This contrucdiction proves the claim.

Chapter 6: Section 8, Number 1

A SOLUTION.

I'll use the notations form the chapter 6.8 in your book.

Since f is continuously differentiable and $\partial^2 f/\partial y \partial x(v)$ exists and is continuous for $v \in U$ the proof in the book shows that for any $(a', a'') \in U$ the limit

$\lim_{(h',h'')\to 0} k(h',h'')$ exists and is equal to $\partial^2 f/\partial y\partial x(a',a'')$ where $k(h',h'') := [f(a'+h',a''+h"")-f(a'+h',a'')-f(a',a''+h"")+f(a',a'')]/h'h''$
In other words for any $\epsilon > 0$ there exists $\delta > 0$ such that for any $(h',h''), h',h'' \neq 0, |h'|,|h''| < \delta$ we have

$$|k(h',h'') - \partial^2 f/\partial x\partial y(a',a'')| < \epsilon$$

This implies (?) that
$\lim_{h'\to 0}\lim_{h''\to 0} k(h',h'')$ exists and is equal to $\partial^2 f/\partial x\partial y(a',a'')$. But

$$\lim_{h''\to 0} k(h',h'') = 1/h'[\partial f(a'+h')/\partial y - \partial f(a')/\partial y]$$

Therefore

$$\lim_{h'\to 0}\lim_{h''\to 0} k(h',h'') = \lim_{h'\to 0} 1/h'[\partial f(a'+h')/\partial y - \partial f(a')/\partial y]$$

So we see that the second partial derivative $\partial^2 f/\partial x\partial y(a',a'')$ exists and is equal to $\partial^2 f/\partial y\partial x(a',a'')$