Since the last lecture covered some difficult material and its presentation in the book [Chapter 5, Section 5] is very sketchy, I have decided to write-up the results in more detail. The same is true for the today's lecture [see Chapter 5, Section 7].

1) Let X be a subset of a normed vector space and $\{x_n\}, n > 0$ be sequence of points in X. We say that a sequence $\{y_m\}, m > 0$ of points in X is a *subsequence* of $\{x_n\}$ if there exists a increasing sequence $n_1 < n_2 < ... < n_m < ...$ of integers such that $y_m = x_{n_m}, m > 0$. [So $y_1 = x_{n_1}$ ...].

Lemma. If $\{y_m\} \in X, m > 0$ is a subsequence of $\{x_n\}$ and $\{z_r\}, r > 0$ is a subsequence of $\{y_m\}$ then $\{z_r\}, r > 0$ is a subsequence of $\{x_n\}$.

Proof of Lemma.

Since $\{y_m\}, m > 0$ is a subsequence of $\{x_n\}$ we can find an increasing sequence of $n_1 < n_2 < ... < n_m < ...$ of integers such that $y_m = x_{n_m}, m > 0$. Since $\{z_r\}, r > 0$ is a subsequence of $\{y_m\}$ we can find an increasing sequence of $m_1 < m_2 < ... < m_r < ...$ of integers such that $z_r = y_{m_r}, r > 0$. Consider now the sequence $\tilde{n}_r$ of integers where $\tilde{n}_r := n_{m_r}$. It is clear that $\tilde{n}_r$ is an increasing sequence of integers and $z_r = x_{\tilde{n}_r}, r > 0$. The lemma is proved.

2)Let $(V, \|v\|_V)$ be a normed vector space, $X \subset V$. We say that X is *compact* if for any sequence $\{x_n\} \in X, n > 0$ we can find a subsequence $\{y_m\}$ of $\{x_n\}$ and a point $x_0 \in X$ such that the subsequence $\{y_m\}$ is convergent to x_0.

3)Let $(V', \|v'\|_{V'}), (V'', \|v''\|_{V''})$ be normed vector spaces. Let V be the *direct product* $V' \times V''$ [see Chapter 2, Section 2 Problem 7 in Corwin, Szczarba]. We denote by $\|v\|_{max}$ the norm on V such that $\|(v', v'')\|_{max} = max(\|v'\|_{V'}, \|v''\|_{V''})$.

Check that $(V, \|v\|_{max})$ is a normed vector space!

For any subsets $X' \subset V', X'' \subset V''$ define a subset $X := X' \times X'' \subset V' \times V''$ as the set of pairs $x = (x', x'')$ where $x' \in X, x'' \in X''$.

Theorem 1. For any pair of compact subsets $X' \subset', X'' \subset V''$ the set $X := X' \times X''$ is compact.

Proof. Let $\{x_n\} \in X, n > 0$ be any sequence. We can write $x_n = (x'_n, x''_n), x'_n \in X, x''_n \in X''$. Since the set X' is compact there exists an increasing sequence $n_1 < n_2 < ... < n_m < ...$ of integers and a point $x'_0 \in X'$ such that the subsequence $y'_m := x'_{n_m}, m > 0$ is convergent to x'_0.

Consider now the sequence $\{y_m\} \in X, y_m := x_{n_m}, m > 0$. Then $\{y_m\}$ is a subsequence of the sequence $\{x_n\}, y_m = (y'_m, y''_m), y'_m \in X', y''_m \in X''$ and the sequence $\{y'_m\}$ is convergent to $x'_0 \in X'$.

Consider now the sequence $\{y''_m\}, m > 0$. Since the set X'' is compact there exists an increasing sequence $m_1 < m_2 < ... < m_r < ...$ of integers and a point $x''_0 \in X''$ such that the subsequence $\{z''_r\}, z''_r := y''_{m_r}, r > 0$ of the sequence $\{y''_m\}$ is convergent to x''_0. As follows from the proof of Lemma there exists an increasing sequence $\tilde{n}_r, r > 0$ of integers such that $z''_r := x''_{\tilde{n}_r}, r > 0$. Consider the sequence $\{z_r\}, z_r := x_{\tilde{n}_r} \in X, r > 0$. Then $z_r = (z'_r, z''_r), z'_r \in X', z''_r \in X'''$. By the construction the sequence z''_r is convergent to $x''_0 \in X''$. On the other hand, by the construction the sequence z'_r is a subsequence of a convergent sequence y'_m. Therefore the sequence z'_r is is convergent to $x'_0 \in X'$. So the sequence $\{z_r\}$ is is convergent to $x_0 := (x'_0, x''_0) \in X$ in $(V, \|v\|_{max})$. Since, by the construction the sequence $\{z_r\}$ is a subsequence of the sequence $\{x_n\}$ we see that any sequence $\{x_n\} \in X, n > 0$ has a convergent subsequence. Theorem 1 is proved.

Theorem 2. Let f be a continuous function on a closed interval $[a, b]$ such that $f(a) < 0, f(b) > 0$. Then there exists a point $c \in [a, b]$ such that $f(c) = 0$.

Proof of the Theorem 2. Let $L \subset \mathbb{R}$ be the half-line $L = \{y \in \mathbb{R} | y \leq 0\}$ and $X := f^{-1}(L) \subset [a, b]$ be the preimage of Y (that is $X = \{x \in [a, b] | f(x) \leq 0\}$). Since f is continuous and Y is closed we know (?) that the subset $X \subset [a, b]$ is closed. Since $f(a) < 0$ we see that $a \in X$ and therefore X is not empty. Since $X \subset [a, b]$ we know that X is bounded. Therefore we can define $c := lub(X)$ (where lub denotes the least upper bound). Since X is closed we know (?) that $c \in X$. Therefore $f(c) \leq 0$. I claim that $f(c) = 0$. To prove that $f(c) = 0$ we show that the assumption that $f(c) \neq 0$ leads to a contradiction.

So assume that $f(c) \neq 0$. Since we know that $f(c) \leq 0$ the assumption that $f(c) \neq 0$ implies that $f(c) < 0$. So $f(c) = -\epsilon, \epsilon > 0$. Since f is continuous we can find $\delta > 0$ such that for all $x \in [a, b], |x - c| < \delta$ we have $|f(x) - f(c)| < \epsilon$. In particular, $|f(c + \delta/2) - f(c)| < \epsilon$ and therefore $f(c + \delta/2) < f(c) + \epsilon = 0$. So $c + \delta/2 \in X$. But we have defined c as the least upper bound of X. CONTRADICTION. Theorem 2 is proved.