MATH 23A
SOLUTION SET 2

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Themes:

- Theme 1 of assignment 2: Give an explicit example when you need to show something exists (such as a counterexample). (See 2.4.5(a), 2.5.3, 2.6.5(b).)
- Theme 2 of assignment 2: Proofs using symbols are usually the cleanest. (See 2.4.6(c), 2.4.6(d), 2.6.4(b).)

2.4.5:

(a) We want to investigate the solutions for a_1, a_2, a_3 in the equation

$$a_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + a_3 \begin{bmatrix} 0 \\ 1 \\ \alpha \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We get three equations:

$$\begin{aligned} a_1 + a_2 &= 0 \\ a_1 + 2a_2 + a_3 &= 0 \\ a_2 + \alpha a_3 &= 0. \end{aligned}$$

The first two yield $a_1 = a_3$ and $a_2 = -a_3$, so the third equation becomes $0 = -a_3 + \alpha a_3 = (\alpha - 1)a_3$. At this point, many of you said $\alpha - 1 = 0$ and $\alpha = 1$, without giving too much more justification. The reasoning should proceed as follows.
Case 1: $\alpha \neq 1$. Then $\alpha - 1 \neq 0$, and a_3 must be 0. But then $a_1 = 0$ and $a_2 = 0$. Hence the only linear combination of the three given vectors that gives 0 is the trivial linear combination. This means that the vectors are linearly independent.
Case 2: $\alpha = 1$. Then $(a_1, a_2, a_3) = (1, -1, 1)$ is a solution, so

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ \alpha \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and the vectors are linearly dependent.
Therefore, the vectors are linearly dependent if and only if $\alpha = 1$.
When you are asked to show the linear dependence of some vectors, it is safest to provide some nontrivial linear combination of them that equals the zero vector.

(b) Note that this problem should have asked for a plane *through the origin*, because any three points in $\mathbb{R}^3$ are in a plane.
The only value of α to consider is 1. The equation of a plane through the origin is $ax + by + cz = 0$, so we just plug in each of the three vectors to get three equations:

$$\begin{aligned} a + b &= 0, \\ a + 2b + c &= 0, \\ b + c &= 0. \end{aligned}$$

Solving, we get $a = c$ and $b = -c$. The equation $cx - cy + cz = 0$ gives the same plane for any nonzero c, so pick $c = 1$ and get the equation $x - y + z = 0$ (other answers are possible). The three vectors automatically satisfy this equation because that was how we got a, b, and c.

2.4.6:

(a) $v_1, \ldots, v_k$ are linearly independent if and only if the only time $c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0$ is when $c_1 = c_2 = \cdots = c_k = 0$. This is equivalent to saying that for any vector $w \in \mathbb{R}^n$, there is at most one way of writing w as a linear combination of $v_1, \ldots, v_k$. In different situations, one formulation might be easier to use than the other.

$v_1, \ldots, v_k$ span $\mathbb{R}^n$ if and only if every vector $w \in R^n$ can be written as a linear combination of $v_1, \ldots, v_k$.

$v_1, \ldots, v_k$ form a basis of $\mathbb{R}^n$ if and only if they are linearly independent and span $\mathbb{R}^n$.

(b) $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $A^2 = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$, and $A^3 = \begin{bmatrix} 13 & 14 \\ 14 & 13 \end{bmatrix}$. We can regard 2×2 matrices as 4×1 vectors in $\mathbb{R}^4$ by considering a matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ as the vector $\begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{bmatrix}$. Then every linear combination $c_1I + c_2A + c_3A^2 + c_4A^3$ of I, A, A^2, and A^3 can be regarded as

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 5 \\ 4 \\ 4 \\ 5 \end{bmatrix} + c_4 \begin{bmatrix} 13 \\ 14 \\ 14 \\ 13 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 5 & 13 \\ 0 & 2 & 4 & 14 \\ 0 & 2 & 4 & 14 \\ 1 & 1 & 5 & 13 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}.$$

Hence the span of I, A, A^2, and A^3 is the image of this 4×4 matrix; the dimension of the span is 2. Elements of the kernel correspond to linear combinations that give the zero element. We can get many different nonzero elements in the kernel, such as $\begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix}$. This means that $-3I - 2A + A^2 + 0A^3 = 0$ and that I, A, A^2, A^3 are not linearly independent.

If you did not see this connection between $\text{Mat}(2, 2)$ and $\mathbb{R}^4$, it is perfectly okay. It is acceptable to prove that I and A form a basis for V, but then you need to prove that they are linearly independent and span V. One way to show they span V: Note that $A^2 = 2A + 3I$ and $A^3 = 7A + 6I$. So if an element B is in V, then for some c_1, c_2, c_3, c_4, $B = c_1I + c_2A + c_3A^2 + c_4A^3 = c_1I + c_2A + c_3(2A + 3I) + c_4(7A + 6I) = (c_1 + 3c_3 + 6c_4)I + (c_2 + 2c_3 + 7c_4)A$. Hence any element in V is a linear combination of I and A.

(c) A good way to start is to try to figure out what W looks like. So let us take a matrix $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then B is in W if and only if $AB = BA$, which is the same as

$$\begin{bmatrix} a+2c & b+2d \\ 2a+c & 2b+d \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} a+2b & 2a+b \\ c+2d & 2c+d \end{bmatrix}.$$

This is equivalent to the four equations

$$\begin{aligned} a + 2c &= a + 2b \\ b + 2d &= 2a + b \\ 2a + c &= c + 2d \\ 2b + d &= 2c + d. \end{aligned}$$

Solving the equations, we get $a = d$ and $b = c$. Therefore, these are necessary and sufficient conditions for B to be in W. Hence

$$W = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} : a, b \in \mathbb{R} \right\}.$$

In this form, it is easy to show that W is a subspace of $\mathrm{Mat}(2,2)$, and that W has dimension 2 (due to the basis $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$).

It is also possible to show that W is a subspace without characterizing W as above (though it is harder to find W's dimension without characterizing W). To show that W is a subspace of $\mathrm{Mat}(2,2)$, there are three things to show:
(i) W is nonempty. (Almost all of you forgot about this.)
(ii) For any matrices B_1 and B_2 in W, $B_1 + B_2$ is also in W.
(iii) For any real number a and matrix B in W, aB is also in W.
For (i), it suffices to note that the zero matrix satisfies $AB = BA$ and is in W. For (ii), suppose B_1 and B_2 are in W. Then $A(B_1 + B_2) = AB_1 + AB_2 = B_1A + B_2A = (B_1 + B_2)A$ (here we use the fact that B_1 and B_2 are in W). Thus $B_1 + B_2$ is in W by definition. For (iii), suppose a is real and B is in W. Then $A(aB) = a(AB) = a(BA) = B(aA)$ (using the fact that $AB = BA$), so aB is in W by definition.

(d) To show that $V \subset W$, we need to show that any matrix B in V is also in W, i.e. $AB = BA$. Now if B is in V, then B can be written as a linear combination $c_1I + c_2A + c_3A^2 + c_4A^3$. This means that $AB = A(c_1I + c_2A + c_3A^2 + c_4A^3) = A(c_1I) + A(c_2A) + A(c_3A^2) + A(c_4A^3) = c_1A + c_2A^2 + c_3A^3 + c_4A^4$. On the other hand, $BA = (c_1I + c_2A + c_3A^2 + c_4A^3)A = (c_1I)A + (c_2A)A + (c_3A^2)A + (c_4A^3)A = c_1A + c_2A^2 + c_3A^3 + c_4A^4$. Therefore, $AB = BA$ and B is in W. If you have already found a basis for V, it is also possible to do this problem by showing that every basis element of V is in W. (Why does this imply that $V \subset W$?)
The facts that $\dim V = 2 = \dim W$ and $V \subset W$ imply that $V = W$. Alternatively, you can show that $W \subset V$ (for which it suffices to show that each basis element of a basis for W is in V).

2.4.8: The two different kinds of coordinates must specify the same vector, so $xe_1 + xe_2 = uv_1 + vv_2$. This means that $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u+v \\ u+3v \end{bmatrix}$.

- To go from the $\{v_1, v_2\}$ coordinates to the $\{e_1, e_2\}$ coordinates, we start with u and v and must solve for x and y in terms of u and v. The above equation immediately gives $x = u + v$ and $y = u + 3v$. Thus we go from (u, v) to $(u + v, u + 3v)$.
- To go from the $\{e_1, e_2\}$ coordinates to the $\{v_1, v_2\}$ coordinates, we start with x and y and must solve for u and v in terms of x and y. The above vector equation gives $u = \frac{3x-y}{2}$ and $v = \frac{-x+y}{2}$. Thus we go from (x, y) to $(\frac{3x-y}{2}, \frac{-x+y}{2})$.

The vector $\begin{bmatrix} 3 \\ -5 \end{bmatrix}$ is just $(3, -5)$ in the $\{e_1, e_2\}$ coordinates (i.e. x and y), so we use the second set of equations above to get $(\frac{3(3)-(-5)}{2}, \frac{-3+(-5)}{2}) = (7, -4)$ in the $\{v_1, v_2\}$ coordinates.

2.4.9:

(a) First suppose that $v_1, \ldots, v_n$ are linearly independent. Recall that a function is one-to-one when different elements in the domain get mapped to different elements in the range. So suppose $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ and $\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ are distinct elements in $\mathbb{R}^n$. They get mapped to $P_{\{v\}} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \sum a_i v_i$ and $P_{\{v\}} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \sum b_i v_i$. If $\sum a_i v_i = \sum b_i v_i$, then they represent two distinct linear combination representations of the same element. But this cannot happen by the linear independence of $v_1, \ldots, v_n$. Therefore, $\sum a_i v_i \neq \sum b_i v_i$, and indeed different elements get mapped to different elements. This means that $P_{\{v\}}$ is one-to-one.

Next suppose that $P_{\{v\}}$ is one-to-one. One way to prove that $v_1, \ldots, v_n$ are linearly independent is to show that for any w in $\mathbb{R}^m$, w can be written as a linear combination of $v_1, \ldots, v_n$ in at most one way. We proceed by contradiction and suppose that this is not true. Then there exists a w in $\mathbb{R}^m$ that can be written in two different ways as a linear combation, say as $\sum a_i v_i$ and $\sum b_i v_i$. But $\sum a_i v_i = P_{\{v\}} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ and $\sum b_i v_i = P_{\{v\}} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$, so the two different vectors $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ and $\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ get mapped to the same element by $P_{\{v\}}$, contradicting the fact that $P_{\{v\}}$ is one-to-one. Therefore, $v_1, \ldots, v_n$ are indeed linearly independent.

(b) First suppose that $v_1, \ldots, v_n$ span $\mathbb{R}^m$. Then for any w in $\mathbb{R}^m$, it is possible to write w as $\sum c_i v_i$ for some $c_1, \ldots, c_n$. But then $w = \sum c_i v_i = P_{\{v\}} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$. What we just showed is that any element in $\mathbb{R}^m$ is in the image of $P_{\{v\}}$. Therefore, $P_{\{v\}}$ is onto.

Next suppose that $P_{\{v\}}$ is onto. Then for any w in $\mathbb{R}^m$, there exists element $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ in $\mathbb{R}^n$ such that $w = P_{\{v\}} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$. But this means exactly that $w = \sum c_i v_i$ and that w is in the span of $v_1, \ldots, v_n$. Because w is an arbitrary element in $\mathbb{R}^m$, we have just shown that $v_1, \ldots, v_n$ span $\mathbb{R}^m$.

Note that it is possible to avoid a two-part proof by going from "$v_1, \ldots, v_n$ span $\mathbb{R}^m$" to "$P_{\{v\}}$ is onto" by a series of if and only if statements. However, this method requires *all* deductions to be if and only if, and it often gets confusing which things you are assuming. Hence it is usually best to write out the two directions separately.

(c) Here is an example of a proof that avoids being two-part. Note that it is extremely short; long proofs are difficult to do in one part.

$$\begin{aligned} v_1, \dots, v_n \text{ is a basis of } \mathbb{R}^m &\iff v_1, \dots, v_n \text{ are linearly independent and } v_1, \dots, v_n \text{span} \mathbb{R}^m \\ &\iff P_{\{v\}} \text{ is one-to-one and } v_1, \dots, v_n \text{ span } \mathbb{R}^m \text{(by part (a))} \\ &\iff P_{\{v\}} \text{ is one-to-one and } P_{\{v\}} \text{ is onto (by part (b)).} \end{aligned}$$

2.5.2:

(a) A basis for the kernel is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$. A basis for the image is $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$. Note that many solutions are possible for all parts of this problem. Feel free to ask a CA any questions you have about row reduction, but row reduction should not be necessary for the remainder of the course.

(b) A basis for the kernel is $\left\{ \begin{bmatrix} -\frac{1}{3} \\ -\frac{4}{3} \\ 1 \end{bmatrix} \right\}$. A basis for the image is $\left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} \right\}$.

(c) A basis for the kernel is $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$. A basis for the image is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$.

2.5.3: False. Many of you saw that $\text{Img}\, g \subset \ker f$ by noting that $f(g(x)) = (f \circ g)(x) = 0$ for any x and hence for any element $g(x)$ in $\text{Img}\, g$. However, showing that f is 0 on $\text{Img}\, g$ is *not* the same thing as showing that f is 0 *only* on $\text{Img}\, g$. Many of you noted this and stopped. This is not good enough, because sometimes the two things *are* the same! For example, suppose $m = 0$ and that the domain of f is just $\{0\}$. Then $\text{Img}\, g$ and $\ker f$ are both always $\{0\}$ (check this).

Therefore, to disprove something, you must give an explicit counterexample. Many are possible for this problem. One is to let $n = m = k = 1$ and let f and g be the constant 0 map, mapping everything to 0. Clearly $f \circ g = 0$. But $\text{Img}\, g = \{0\}$ and $\ker f = \mathbb{R}^1 \neq \{0\}$.

2.5.4:

(a) Some of you were confused by the identification of P_2 as $\mathbb{R}^3$. What it means practically is that the coefficients of a polynomial "represent" the polynomial in some ways. However, polynomials are still polynomials—the identification as $\mathbb{R}^3$ is just a convenient way of looking at things. A similar situation is in 2.4.6(b) where $\text{Mat}(2,2)$ is identified as $\mathbb{R}^4$ but is still in actuality the set of 2×2 matrices.
On to the problem. We see what T does to a general polynomial $a + bx + cx^2$ in P_2:

$$\begin{aligned} T(\text{the polynomial } a + bx + cx^2) &= x(\text{the derivative of the polynomial } a + bx + cx^2) \\ &\quad + x^2(\text{the second derivative of the polynomial } a + bx + cx^2) \\ &= x(b + 2cx) + x^2(2c) \\ &= bx + 4cx^2. \end{aligned}$$

(You do not have to write the text in the above equations; they are just there for explanatory purposes.) In the same way that we think of $a + bx + cx^2$ as $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, we can think of $bx + 4cx^2$ as $\begin{bmatrix} 0 \\ b \\ 4c \end{bmatrix}$. The array for T is just the 3×3 array A for which $A \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ b \\ 4c \end{bmatrix}$. The answer is $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

(b) If we use the above matrix, then a basis for the image is $\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} \}$, which really *means* $\{x, 4x^2\}$. A basis for the kernel is $\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \}$, which really *means* $\{1\}$. Different answers are possible for this problem.

2.5.7: A basis for the image of A is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

A basis for the kernel of A is

$$\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{10}{3} \\ -\frac{8}{3} \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Thus $\dim(\text{Img}) + \dim(\ker) = 2 + 3 = 5$, which is the dimension of the domain, $\mathbb{R}^5$.

A basis for the image of B is

$$\left\{ \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}.$$

A basis for the kernel of B is

$$\left\{ \begin{bmatrix} -\frac{3}{4} \\ -\frac{3}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{5}{2} \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{3}{4} \\ -\frac{1}{2} \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Thus $\dim(\text{Img}) + \dim(\ker) = 2 + 3 = 5$, which is the dimension of the domain, $\mathbb{R}^5$.

Many answers are possible for both parts.

2.5.9: This is probably the most challenging problem on the set, so please do not be discouraged if you did not solve it. One hint in the probem is that you need to show the unique existence of something. This often means that there is a bijective map around, because for any element y in the range of the map, there exists a unique element x in the domain that is mapped to y. We now define this map. Let P_{2k-1} be the set of polynomials of degree at most $2k-1$. Note that P_{2k-1} is a $2k$-dimensional vector space with basis $\{1, x, x^2, \ldots, x^{2k-1}\}$. Now we define map $T\colon P_{2k-1} \to \mathbb{R}^{2k}$ by $T(p) = \begin{bmatrix} p(1) \\ p(2) \\ \vdots \\ p(k) \\ p'(1) \\ p'(2) \\ \vdots \\ p'(k) \end{bmatrix}$. Because the derivative is a linear map, it can be shown that T is linear. We are almost done, because if we can show that T is bijective, then for any $a_1, \ldots, a_k, b_1, \ldots, b_k$, there exists a unique p in P_{2k-1} such that $T(p) = \begin{bmatrix} a_1 \\ \vdots \\ a_k \\ b_1 \\ \vdots \\ b_k \end{bmatrix}$, i.e., $p(i) = a_i$ and $p'(i) = b_i$ for $i = 1, \ldots, k$.

Because the domain and range of T have the same dimension, T is bijective if and only if it is one-to-one, which occurs if and only if the kernel of T is $\{0\}$ (check this). So we have boiled down the problem into showing that 0 is the only element in the kernel. Suppose p is in the kernel. Then $T(p) = 0$, so $0 = p(1) = p(2) = \cdots = p(k) = p'(1) = p'(2) = \cdots = p'(k)$. By the hint in the book, $p(x) = (x-1)^2 q_1(x)$ for some polynomial q_1. Now $q_1(i) = q_1'(i) = 0$ for $i = 2, 3, \ldots, k$ because $p(i) = p'(i) = 0$ for $i = 2, 3, \ldots, k$ (why does it follow?). So by the hint, $q_1(x) = (x-2)^2 q_2(x)$ for some polynomial q_2. Hence $p(x) = (x-1)^2(x-2)^2 q_2(x)$. We can keep doing this and eventually get $p(x) = (x-1)^2(x-2)^2 \ldots (x-k)^2 q_k(x)$. But the right hand side is at least degree $2k$ if $q_k \neq 0$. Because $\deg(p(x)) < 2k$, this must mean that $q_k = 0$. Hence $p = 0$, and the only element that can be in the kernel is 0.

2.6.4:

(a) A lot of you confused $\mathcal{L}(\text{Mat}(n,n), \text{Mat}(n,n))$ with $\text{Mat}(n,n)$. In this problem, $n \times n$ matrices are being regarded as *vectors*, and L contains *maps* that take matrix *vectors* to matrix *vectors*. Like in 2.4.6(b), we can regard $\text{Mat}(n,n)$ as $\mathbb{R}^{n^2}$ ($\text{Mat}(n,n)$ just has the coordinates arranged in a square). Hence we are really dealing with linear functions from $\mathbb{R}^{n^2}$ to $\mathbb{R}^{n^2}$. From this it is easy to check that $\mathcal{L}(\text{Mat}(n,n), \text{Mat}(n,n))$ is a vector space, so I will not do it here. One final observation is that linear maps from $\mathbb{R}^{n^2}$ to $\mathbb{R}^{n^2}$ correspond to $n^2 \times n^2$ matrices. Hence the dimension of L is $(n^2)(n^2) = n^4$.

(b) Let A be any given $n \times n$ matrix. To show that L_A is linear, we need to show that
- for any B_1 and B_2 in $\text{Mat}(n,n)$, $L_A(B_1 + B_2) = L_A(B_1) + L_A(B_2)$
- for any real number c and any B in $\text{Mat}(n,n)$, $L_A(cB) = cL_A(B)$.

For the former, $L_A(B_1 + B_2) = A(B_1 + B_2) = AB_1 + AB_2 = L_A(B_1) + L_A(B_2)$. For the latter, $L_A(cB) = A(cB) = cAB = cL_A(B)$. Hence L_A is linear. The proof of the linearity of R_A is similar.

It is also correct to do this problem by comparing the entries of both sides of the equations.

(c) Many of you wrote that $\{L_A : A \in \mathrm{Mat}(n, n)\}$ has dimension n^2 because it corresponds to the n^2-dimensional vector space $\mathrm{Mat}(n, n)$. However, you also need to show that correspondence between the two spaces is linear. Otherwise, it is conceivable that the two spaces have different dimensions. (For example, there exists non-linear bijective maps from $\mathbb{R}^2$ to $\mathbb{R}$.) Here is how the proof can be made rigorous. Denote $W = \{L_A : A \in \mathrm{Mat}(n, n)\}$. Then we can define a map $T\colon \mathrm{Mat}(n, n) \to W$ by $T(A) = L_A$. We must then check the following:

- T is linear.
- T is one-to-one.
- T is onto.

It is not difficult to show that T is linear. It is obvious that T is onto (by how W is defined). The only hard part is to show that T is one-to-one. Recall that this is equivalent to showing that $\ker(T) = \{0\}$. So given a nonzero $n \times n$ matrix A, we must show that $T(A) = L_A$ is nonzero, meaning that L_A is not the zero map. Let I be the $n \times n$ identity matrix. Then $L_A(I) = AI = A$, which is nonzero by assumption. Hence L_A is not the zero map, and indeed $\ker(T) = \{0\}$.

A similar proof is used to show that the space of transformations of the form R_A also has dimension n^2.

(d) We try to find the general form of all transformations of the form $L_A + R_B$. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$. Then $L_A + R_B$ takes

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \text{ to } \begin{bmatrix} (a+e)x_{11} + gx_{12} + bx_{21} & fx_{11} + (a+h)x_{12} + bx_{22} \\ cx_{11} + (d+e)x_{21} + gx_{22} & cx_{12} + fx_{21} + (d+h)x_{22} \end{bmatrix}.$$

Let us now regard 2×2 matrices $\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$ as vectors $\begin{bmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \end{bmatrix}$ in $\mathbb{R}^4$. Then the matrix representation of $L_A + R_B$ is $\begin{bmatrix} a+e & g & b & 0 \\ f & a+h & 0 & b \\ c & 0 & d+e & g \\ 0 & c & f & d+h \end{bmatrix}$. It is clear that no matter what values a, b, c, d, e, f, g, h are, this matrix can never equal $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Hence this represents a linear transformation T_0 that cannot be written as $L_A + R_B$. Explicitly, $T_0 \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} x_{22} & 0 \\ 0 & 0 \end{bmatrix}$.

Many other answers are possible.

2.6.5:

(a) W is a subspace of V if and only if

(i): W is nonempty. (Almost all of you forgot about this.)

(ii): For any vectors w_1 and w_2 in W, $w_1 + w_2$ is also in W.

(iii): For any real number a and vector w in W, aw is also in W.

(b)

(i): Let W_1 denote the set in question. It is not a subspace of V. One way to see this is that W_1 does not satisfy additive closure. First of all, if W_1 is empty then it is automatically not a subspace. But even if W_1 is nonempty (say it contains f), we know that $f + f = 2f$ is not in W_1 because $2f = f + f = (f'(x) + 1) + (f'(x) + 1) = (2f)'(x) + 2$, which cannot possibly be the function $(2f)'(x) + 1$ (just try plugging in $x = 1/2$).

What many of you did contains a subtle fallacy. Many of you wrote that if $f, g \in W_1$ then $f(x) + g(x) = f'(x) + g'(x) + 2 \neq f'(x) + g'(x) + 1$, and hence additive closure is violated. Well, it is only violated if we can find an actual example of a violation. An analogous example is the statement "For any apple that is an orange, the apple likes to shop." It is absurd that any apple likes to shop, but since we cannot find an apple that is an orange, the above statement is actually true.

Hence you need to exhibit some f and g in W_1 (or deduce that some exists, like I did above) in order to have the additive closure violated. The function $f(x) = 1$ is a *specific* function you could have used.

Despite what I just said, it is actually a good technique to see if additive closure is satisfied by using f and g. This way, you discovered that $f + g$ is not in W_1 if $f'(x) + g'(x) + 2$ and $f'(x) + g'(x) + 1$ differ. Often it will not be so obvious why they might differ. Then you need to find specific f and g in W_1 *such that* the two quantities (or in this case functions) differ.

(ii): Let W_2 denote the set in question. It is a subspace of V. Let us check the three criteria of a subspace (see 2.6.5(a) above):

- W_2 is nonempty because it contains the function $f(x) = 0$.
- Suppose $f(x)$ and $g(x)$ are in W_2. Then $(f + g)(x) = f(x) + g(x) = xf'(x) + xg'(x) = x(f'(x) + g'(x)) = x((f + g)'(x))$. Hence $(f + g)(x)$ is also in W_2.
- Suppose a is a real number and $f(x)$ is in W_2. Then $(af)(x) = af(x) = a(xf'(x)) = x(af)'(x)$. Hence $(af)(x)$ is also in W_2.

(iii): Let W_3 denote the set in question. It is not a subspace of V. We note that the function $h(x) = \frac{1}{4}x^2$ is in W_3. However, $(2h)(x)$ turns out not to be in W_3: $((2h)'(x))^2 = (x)^2 = x^2$, while $(2h)(x) = \frac{1}{2}x^2$, a different function from x^2. Hence W_3 violates additive closure.

Many of you wrote that if $f, g \in W_3$ then $((f + g)'(x))^2 = (f'(x) + g'(x))^2 = (f'(x))^2 + f'(x)g'(x) + (g'(x))^2 \neq (f'(x))^2 + (g'(x))^2 = f(x) + g(x)$, and hence additive closure is violated. Well, the $\neq$ becomes $=$ if $f'(x)g'(x)$ is identically 0. This can happen if $f(x) = g(x) = 0$: you can check that such f and g, as well as $f + g$, are in W_3. This is situation where we need to find actual f and g such that $f'(x)g'(x)$ is not identically 0.