MATH 23A PROBLEM SET 3

due October 13th

1. Using the axioms of Vector spaces as in your book [p. 189], prove that $0v = \vec{0} \forall v \in V$.
2. Let $F\colon V \to V'$ be a linear map, $L = \ker F, W = \operatorname{Im} F$. Let $l_1, \dots, l_r$ be vectors in L and $v_1, \dots, v_m$ be vectors in V. For all $i, 1 \le i \le m$ we define $w_i = F(v_i) \in W$.
   (a) Assume that the vectors $\{l_1, \dots, l_r\} \in L$ are linearly independent and that the vectors $\{w_1, \dots, w_m\} \in W$ are linearly independent. Prove that the vectors $\{l_1, \dots, l_r, v_1, \dots, v_m\} \in V$ are also linearly independent.
   (b) Assume that the vectors $\{l_1, \dots, l_r\}$ span L and that the vectors $\{w_1, \dots, w_m\}$ span W. Prove that the vectors $\{l_1, \dots, l_r, v_1, \dots, v_m\}$ span V.
   (c) Assume that the vectors $\{l_1, \dots, l_r\}$ are a basis of L and that the vectors $w_1, \dots, w_m$ are a basis of W. Prove that the vectors $\{l_1, \dots, l_r, v_1, \dots, v_m\}$ are a basis of V.
3. Let V be a two-dimensional vector space, and $\{e_1, e_2\}$ and $\{v_1, v_2\}$ be two bases of V, such that $v_1 = e_1 + e_2, v_2 = e_1 - e_2$. Let e^1, e^2 and v^1, v^2 be the corresponding dual bases of V^*. Find the expression of v^1, v^2 in terms of e^1, e^2.
4. Let $F\colon \mathbb{R}^n \to \mathbb{R}$ be the map given by

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \mapsto \sum_{i=1}^{n} x_i.$$

   Find the dimension of $\ker F$.
5. Hubbard, 2.6.7