MATH 23A SOLUTION SET 3

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1.
$$0v = (0+0)v = 0v + 0v$$
Therefore
$$0v + (-(0v)) = 0v + 0v + (-(0v))$$
and hence
$$0 = 0v$$

2. (a) Suppose
$$a_1 l_1 + \cdots + a_r l_r + b_1 v_1 + \cdots + b_m v_m = 0$$
for some $\{a_i\}$ and $\{b_j\}$. Since $F(0) = 0$, we have
$$F(a_1 l_1 + \cdots + a_r l_r + b_1 v_1 + \cdots + b_m v_m) = 0.$$
Expanding by linearity we have
$$a_1 F(l_1) + \cdots + a_r F(l_r) + b_1 F(v_1) + \cdots + b_m F(v_m) = 0.$$
Since $F(l_i) = 0$ and $F(v_j) = w_j$ this reduces to
$$b_1 w_1 + \cdots + b_m w_m = 0.$$
Since $\{w_i\}$ are linearly independent this means that $b_i = 0$ for all i. That leaves us with
$$a_1 l_1 + \cdots + a_r l_r = 0,$$
but since $\{l_i\}$ are linearly independent, $a_i = 0$ for all i. Therefore $\{l_1, \dots, l_r, v_1, \dots, v_m\}$ are linearly independent.

(b) Given $v \in V$, $F(v) \in W$ so we can write
$$F(v) = a_1 w_1 + \cdots + a_m w_m$$
since $\{w_i\}$ spans W. Therefore
$$F(v - (a_1 v_1 + \cdots + a_m v_m)) = F(v) - (a_1 w_1 + \cdots + a_m w_m) = 0$$
so $v - (a_1 v_1 + \cdots + a_m v_m)$ is in the kernel of F. Since $\{l_i\}$ spans $\ker F$ we can write
$$v - (a_1 v_1 + \cdots + a_m v_m) = b_1 l_1 + \cdots + b_r l_r$$
and hence
$$v = a_1 v_1 + \cdots + a_m v_m + b_1 l_1 + \cdots + b_r l_r$$
so v is in the span of $\{l_1, \dots, l_r, v_1, \dots, v_m\}$.

(c) $\{l_1, \dots, l_r\}$ a basis for L means that they are linearly independent and span L and $\{w_1, \dots, w_r\}$ a basis for W means that they are linearly independent and span W. Combining parts (a) and (b) we clearly have that $\{l_1, \dots, l_r, v_1, \dots, v_m\}$ is a basis of V.

3. Claim: $v^1 = \frac{e^1+e^2}{2}$ and $v^2 = \frac{e^1-e^2}{2}$

Proof. It is clear that v^1 and v^2 are indeed elements of V^* so we just check that they have the desired values on v_1 and v_2.

$$\begin{aligned} v^1(v_1) &= \frac{e^1+e^2}{2}(e_1+e_2) \\ &= \frac{e^1(e_1+e_2)+e^2(e_1+e_2)}{2} \\ &= \frac{(1+0)+(0+1)}{2} \\ &= 1. \end{aligned}$$

Likewise,

$$\begin{aligned} v^1(v_2) &= \frac{e^1+e^2}{2}(e_1-e_2) \\ &= \frac{e^1(e_1-e_2)+e^2(e_1-e_2)}{2} \\ &= \frac{(1-0)+(0-1)}{2} \\ &= 0 \end{aligned}$$

The other calculations are similar and hence will not be carried out. □

4. Since $\operatorname{Im} F$ is a subspace of $\mathbb{R}$ its dimension is either 1 or 0. However, it is not 0 because the image of e_1, the first standard basis vector, is 1. Therefore, $\dim(\operatorname{Im} F) = 1$. Since $\dim(\mathbb{R}^n) = n$, the dimension formula gives us $\dim(\ker F) = n - 1$.
5. (a)

$$T(ap_1+p_2) = (x^2+1)(ap_1+p_2)'' - (x)(ap_1+p_2)' + (2)(ap_1+p_2)$$

By well-known propertiesof the derivative we have

$$\begin{aligned} T(ap_1+p_2) &= (x^2+1)(ap_1''+p_2'') - (x)(ap_1'+p_2') + (2)(ap_1+p_2) \\ &= (x^2+1)(ap_1'') + (x^2+1)p_2'' - (x)(ap_1') - (x)p_2' + (2)(ap_1) + (2)p_2 \\ &= a((x^2+1)p_1'' - (x)p_1' + (2)p_1) + (x^2+1)p_2'' - (x)p_2' + (2)p_2 \\ &= a(T(p_1)) + T(p_2) \end{aligned}$$

(b) We evaluate the map $\Phi_{\{p\}}^{-1} \circ T \circ \Phi_{\{p\}}$ on the standard basis.

$$\begin{aligned} \Phi_{\{p\}}^{-1} \circ T \circ \Phi_{\{p\}}(e_1) &= \Phi_{\{p\}}^{-1} \circ T(1) \\ &= \Phi_{\{p\}}^{-1}(2) \\ &= 2e_1 \end{aligned}$$

$$\begin{aligned} \Phi_{\{p\}}^{-1} \circ T \circ \Phi_{\{p\}}(e_2) &= \Phi_{\{p\}}^{-1} \circ T(x) \\ &= \Phi_{\{p\}}^{-1}(x) \\ &= e_2 \end{aligned}$$

Similarly $\Phi_{\{p\}}^{-1} \circ T \circ \Phi_{\{p\}}(e_3) = 2e_1 + 2e_3$. Thus the columns of the matrix are as desired.

(c) Extending T to an operator from P_3 to P_3 it is clear that the first three columns are the same (with 0s in the last entry) as above. $T(x^3) = (x^2+1)(6x) - x(3x^2) + 2(x^3)$ so the third column is

$$\begin{matrix} 0 \\ 6 \\ 0 \\ 5 \end{matrix}$$

I will now treat the general case. If T is a map from P_n to P_n then we evaluate $T(x^j)$ for all $j \leq n$

$$\begin{aligned} T(x^j) &= (x^2+1)(j(j-1)x^{j-2}) - x(j(x^{j-1})) + 2(x^j) \\ &= (j(j-1) - j + 2)x^j + (j(j-1))x^{j-2} \\ &= (j^2 - 2j + 2)x^j + (j^2 - j)x^{j-2} \end{aligned}$$

Since the k^{th} column is $T(x^{k-1})$, we have the following matrix:

$$\begin{pmatrix} 2 & 0 & 2 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 2 & \ddots & k^2-3k+2 & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & \ddots & 0 \\ 0 & 0 & 0 & \ddots & k^2-4k+5 & \ddots & n^2-n \\ 0 & 0 & 0 & \ddots & 0 & \ddots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & n^2-2n+2 \end{pmatrix}$$