MATH 23A
SOLUTION SET 6

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1.5.4: (a) Let S_n be the nth partial sum of the series $\sum_{k=0}^{\infty} \frac{1}{k!} A^k$. Then

$$\begin{aligned}
|S_n| &= \left| \sum_{k=0}^{n-1} \frac{1}{k!} A^k \right| \\
&\le \sum_{k=0}^{n-1} \left| \frac{1}{k!} A^k \right| && \text{(triangle inequality)} \\
&\le |I| + \sum_{k=1}^{n-1} \frac{1}{k!} |A|^k && (\left|A^k\right| \le |A|^k \text{ when } k \ge 1).
\end{aligned}$$

But $\sum_{k=1}^{n-1} \frac{1}{k!}|A|^k$ is just the $(n-1)$th partial sum of the series $\sum_{k=1}^{\infty} \frac{1}{k!}|A|^k$, which converges to $e^{|A|} - 1$. Hence by the comparison test, the original series $\sum_{k=0}^{\infty} \frac{1}{k!} A^k$ also converges, and $\left|e^A\right| = \left|\sum_{k=0}^{\infty} \frac{1}{k!} A^k\right|$ is bounded above by $|I| + e^{|A|} - 1 = \sqrt{n} + e^{|A|} - 1$.

(b) **(1):** $e^A = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^{\infty} \frac{1}{i!} a^i & 0 \\ 0 & \sum_{j=0}^{\infty} \frac{1}{j!} b^j \end{bmatrix} = \begin{bmatrix} e^a & 0 \\ 0 & e^b \end{bmatrix}$.

(2): $e^A = I + \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \cdots = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$.

(3): $e^A = \begin{bmatrix} 1 - \frac{1}{2!}a^2 + \frac{1}{4!}a^4 - \ldots & \frac{1}{1!}a - \frac{1}{3!}a^3 + \frac{1}{5!}a^5 - \ldots \\ -\frac{1}{1!}a + \frac{1}{3!}a^3 - \frac{1}{5!}a^5 + \ldots & 1 - \frac{1}{2!}a^2 + \frac{1}{4!}a^4 - \ldots \end{bmatrix} = \begin{bmatrix} \cos a & \sin a \\ -\sin a & \cos a \end{bmatrix}$.

(c) **(1):** The condition $AB = BA$ is called "A and B commute." We claim that if A and B commute, then indeed $e^{A+B} = e^A e^B$. Let us see how much $e^A e^B$ differs from the $(r+1)$th partial sum of $\sum_{p=0}^{\infty} \frac{1}{p!}(A+B)^p$. We have

$$\begin{aligned}
e^A e^B - \sum_{p=0}^{r} \frac{1}{p!}(A+B)^p &= \sum_{i=0}^{\infty} \frac{1}{i!} A^i e^B - \sum_{i=0}^{r} A^i \left(\sum_{j=0}^{r-i} \frac{1}{(i+j)!} \binom{i+j}{i} B^j \right) \\
&= \sum_{i=r+1}^{\infty} \frac{1}{i!} A^i e^B + \sum_{i=0}^{r} A^i \left(\frac{1}{i!} e^B - \sum_{j=0}^{r-i} \frac{1}{i!j!} B^j \right) \\
&= \sum_{i=r+1}^{\infty} \frac{1}{i!} A^i e^B + \sum_{i=0}^{r} \frac{1}{i!} A^i \left(e^B - \sum_{j=0}^{r-i} \frac{1}{j!} B^j \right) \\
&= \sum_{i=r+1}^{\infty} \frac{1}{i!} A^i e^B + \sum_{i=0}^{r} \frac{1}{i!} A^i \left(\sum_{j=r-i+1}^{\infty} \frac{1}{j!} B^j \right)
\end{aligned}$$

where the top equality uses the fact that A and B commute (why?). Assume for now r is even. Let $r = 2m$. Then

$$\begin{aligned}
&\left| e^A e^B - \sum_{p=0}^{r} \frac{1}{p!}(A+B)^p \right| \\
&= \left| \sum_{i=2m+1}^{\infty} \frac{1}{i!} A^i e^B + \sum_{i=0}^{2m} \frac{1}{i!} A^i \left(\sum_{j=2m-i+1}^{\infty} \frac{1}{j!} B^j \right) \right| \\
&\leq \left| \sum_{i=2m+1}^{\infty} \frac{1}{i!} A^i e^B \right| + \left| \sum_{i=0}^{2m} \frac{1}{i!} A^i \left(\sum_{j=2m-i+1}^{\infty} \frac{1}{j!} B^j \right) \right| \\
&\leq \sum_{i=2m+1}^{\infty} \left| \frac{1}{i!} A^i e^B \right| + \sum_{i=0}^{2m} \left| \frac{1}{i!} A^i \left(\sum_{j=2m-i+1}^{\infty} \frac{1}{j!} B^j \right) \right| \\
&\leq \sum_{i=2m+1}^{\infty} \left| \frac{1}{i!} A^i \right| \left| e^B \right| + \sum_{i=m}^{2m-1} \left| \frac{1}{i!} A^i \left(\sum_{j=2m-i+1}^{\infty} \frac{1}{j!} B^j \right) \right| + \sum_{i=0}^{m-1} \left| \frac{1}{i!} A^i \left(\sum_{j=2m-i+1}^{\infty} \frac{1}{j!} B^j \right) \right| \\
&\leq \sum_{i=2m+1}^{\infty} \left| \frac{1}{i!} A^i \right| (e^{|B|} + \sqrt{n} - 1) + \sum_{i=m}^{2m-1} \left| \frac{1}{i!} A^i \right| \left(\sum_{j=2m-i+1}^{\infty} \left| \frac{1}{j!} B^j \right| \right) + \sum_{i=0}^{m-1} \left| \frac{1}{i!} A^i \right| \left(\sum_{j=2m-i+1}^{\infty} \left| \frac{1}{j!} B^j \right| \right) \\
&\leq \sum_{i=2m+1}^{\infty} \frac{1}{i!} \left| A^i \right| (e^{|B|} + \sqrt{n} - 1) + \sum_{i=m}^{2m-1} \frac{1}{i!} |A|^i \left(\sum_{j=2m-i+1}^{\infty} \frac{1}{j!} |B|^j \right) + \sum_{i=0}^{m-1} \frac{1}{i!} |A|^i \left(\sum_{j=m}^{\infty} \frac{1}{j!} |B|^j \right) \\
&= \sum_{i=m}^{\infty} \frac{1}{i!} |A|^i (e^{|B|} + \sqrt{n} - 1) + \left(\sum_{i=0}^{m-1} \frac{1}{i!} |A|^i \right) \left(\sum_{j=m}^{\infty} \frac{1}{j!} |B|^j \right) \\
&\leq \sum_{i=m}^{\infty} \frac{1}{i!} |A|^i (e^{|B|} + \sqrt{n} - 1) + \left(\sum_{i=0}^{\infty} \frac{1}{i!} |A|^i \right) \left(\sum_{j=m}^{\infty} \frac{1}{j!} |B|^j \right) \\
&\leq \sum_{i=m}^{\infty} \frac{1}{i!} |A|^i (e^{|B|} + \sqrt{n} - 1) + (e^{|A|} + \sqrt{n} - 1) \left(\sum_{j=m}^{\infty} \frac{1}{j!} |B|^j \right)
\end{aligned}$$

Note that in the fifth inequality above, we used the fact that when $0 \leq i \leq m-1, 2m-i+1 > m$.
The last expression above tends to 0 as m tends to infinity. But let us be more precise. Suppose we are given $\epsilon > 0$. Let $\epsilon_1 = \frac{\epsilon}{2(e^{|B|}+\sqrt{n}-1)} > 0$ and $\epsilon_2 = \frac{\epsilon}{2(e^{|A|}+\sqrt{n}-1)} > 0$. Then there exist M_1 and M_2 such that

$$\sum_{i=m}^{\infty} \frac{1}{i!} |A|^i < \epsilon_1 \text{ when } m \geq M_1, \text{ and}$$

$$\sum_{j=m}^{\infty} \frac{1}{j!} |B|^j < \epsilon_2 \text{ when } m \geq M_2.$$

Let $M = \max\{M_1, M_2\}$. Then whenever $m \geq M$, $\left|e^A e^B - \sum_{p=0}^{2m} \frac{1}{p!}(A+B)^p\right| < \epsilon_1(e^{|B|} + \sqrt{n} - 1) + \epsilon_2(e^{|A|} + \sqrt{n} - 1) = \epsilon$. This only takes care of the case that r is even. However, it shows that the sequence of partial sums

$$\left\{\sum_{p=0}^{2} \frac{1}{p!}(A+B)^p, \sum_{p=0}^{4} \frac{1}{p!}(A+B)^p, \sum_{p=0}^{6} \frac{1}{p!}(A+B)^p, \ldots\right\}$$

converges to $e^A e^B$. The above sequence is a subsequence of

$$\left\{\sum_{p=0}^{1} \frac{1}{p!}(A+B)^p, \sum_{p=0}^{2} \frac{1}{p!}(A+B)^p, \sum_{p=0}^{3} \frac{1}{p!}(A+B)^p, \ldots\right\},$$

which converges to e^{A+B}. Hence by problem 1.5.17, the sequence of even partial sums converges to e^{A+B}, implying that $e^A e^B = e^{A+B}$ (if a sequence converges, it converges to an unique value).

(2): Because A commutes with itself, by (c)(1) we see that $e^{2A} = e^{A+A} = e^A e^A = (e^A)^2$.

Remark. The exponential of a matrix is useful because it turns out to solve *matrix* differential equations. Consider an $n \times n$ matrix $F(t)$, each of whose entries is a real-valued function of t. We can take the derivative $F'(t)$ by differentiating each entry. Then there is a theorem that given constant $n \times n$ matrices A and B, the only $F(t)$ that satisfies

$$F'(t) = AF(t) \text{ and } F(0) = B$$

is $F(t) = e^{tA}B$. For instance, if $A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then $e^{tA}B = e^{tA} = \begin{bmatrix} \cos(ta) & \sin(ta) \\ -\sin(ta) & \cos(ta) \end{bmatrix}$.

1.5.11: Suppose $\bar{X}$ is not closed. Let $Y = \mathbb{R}^n - \bar{X}$. Then Y is not open, so there exists $y \in Y$ such that for any $\epsilon > 0$, $B_\epsilon(y)$ is not contained in Y. By letting $\epsilon = \frac{1}{n}$, we see that for any positive integer n, there exists $\bar{x}_n \in B_{\frac{1}{n}}(y)$ not contained in Y. But this means $\bar{x}_n \in \bar{X}$, so there exists a sequence $x_{n1}, x_{n2}, \ldots$ of points in X that converges to x_n. Due to convergence, there exists $x_n \in \{x_{n1}, x_{n2}, \ldots\}$ such that $|x_n - \bar{x}_n| < \frac{1}{n}$. Now $|\bar{x}_n - y| < \frac{1}{n}$ because $\bar{x}_n \in B_{\frac{1}{n}}(y)$. Hence by the triangle inequality,

$$|x_n - y| \leq |x_n - \bar{x}_n| + |\bar{x}_n - y| < \frac{2}{n}.$$

We claim that the sequence $x_1, x_2, \ldots$ converges to y. For a given $\epsilon > 0$, we can find N such that $\frac{2}{N} < \epsilon$. Then for any $n \geq N$, $|x_n - y| < \frac{2}{n} \leq \frac{2}{N} < \epsilon$. By definition, the sequence $x_1, x_2, \ldots$ converges to y. Because the x_i's are all in X, y is in $\bar{X}$, a contradiction. Therefore, $\bar{X}$ is closed.

1.5.12: (a) First suppose the sequence $a_1, a_2, \ldots$ converges to a. Then for any $\epsilon' > 0$, there exists N such that for $n > N$, $|a_n - a| < \epsilon'$. Now, for a given $\epsilon > 0$, we apply the above to $\epsilon' = \varphi(\epsilon)$ (which is positive) to obtain an N such that for $n > N$, $|a_n - a| < \varphi(\epsilon)$. This is exactly what we want to prove.

Next, suppose that for any $\epsilon' > 0$, there exists N such that for $n > N$, we have $|a_n - a| \le \varphi(\epsilon')$. Fix an $\epsilon > 0$ (e.g. the ϵ our opponent has given us). Because $\lim_{\epsilon' \to 0} \varphi(\epsilon') = 0$, there exists $\epsilon' > 0$ such that $\varphi(\epsilon') < \epsilon$. For this ϵ' (which depends on ϵ), by assumption we can find an N (which now also depends on ϵ) such that for $n > N$, we have $|a_n - a| \le \varphi(\epsilon')$. Since $\varphi(\epsilon') < \epsilon$, we know that when $n > N$, $|a_n - a| < \epsilon$. This is exactly the definition of convergence, so the sequence $a_1, a_2, \dots$ converges to a.

(b) Let φ be as in part (a). Then a function f converges at x_0 iff for $\epsilon > 0$, there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varphi(\epsilon)$.

1.5.14: **(a):** Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ and $g\colon \mathbb{R}^2 \to \mathbb{R}$ be given by $f\binom{x}{y} = x$ and $g\binom{x}{y} = y$. Our problem is to determine whether $\frac{f^2}{f+g}$ has a limit at $\binom{1}{2}$. By corollary 1.5.28, f and g are continuous. Theorem 1.5.27(a) and (d) tell us that f^2 and $f+g$ are continuous. Since $(f+g)\binom{1}{2} = 1+2 = 3 \ne 0$, theorem 1.5.27(c) tells us that $\frac{f^2}{f+g}$ is continuous at $\binom{1}{2}$. Hence the given limit exists and is equal to $(\frac{f^2}{f+g})\binom{1}{2} = \frac{1}{3}$.

(g): This limit does *not* exist because we can find a sequence $\binom{x_1}{y_1}, \binom{x_2}{y_2}, \dots \to \binom{0}{0}$ such that the sequence $(x_1{}^2 + y_1{}^2)(\log|x_1y_1|), (x_2{}^2 + y_2{}^2)(\log|x_2y_2|), \dots$ does not converge. Let $x_n = \frac{1}{n}$ and $y_n = e^{1/x_n{}^3} = e^{-n^3}$. Because x_n and y_n both converge to 0 as $n \to \infty$, $\binom{x_n}{y_n}$ converges to $\binom{0}{0}$. However,

$$\begin{aligned}
(x_n{}^2 + y_n{}^2)(\log|x_ny_n|) &= \left(\frac{1}{n^2} + y_n{}^2\right)(-n^3 + \log x_n) \\
&\le \left(\frac{1}{n^2} + y_n{}^2\right)(-n^3) && \text{since } \left(\frac{1}{n^2} + y_n{}^2\right)\log x_n \le 0 \\
&< \frac{1}{n^2}(-n^3) && \text{since } -y_n{}^2n^3 < 0 \\
&= -n.
\end{aligned}$$

Hence $(x_n{}^2 + y_n{}^2)(\log|x_ny_n|)$ goes to $-\infty$ and diverges.

Note that if we approach $\binom{0}{0}$ from any particular straight line, our limit will be 0. The $\binom{x_n}{y_n}$ we defined above hence must be approaching $\binom{0}{0}$ "from a curve."

(h): We claim that the limit exists and is 0. We will show that for any sequence $\binom{x_1}{y_1}, \binom{x_2}{y_2}, \dots$ converging to $\binom{0}{0}$, the sequence $\{(x_n{}^2+y_n{}^2)\log(x_n{}^2+y_n{}^2)\}$ converges to 0. Suppose we have such sequence $\{\binom{x_n}{y_n}\}$. Because any polynomial function is continuous (corollary 1.5.28), the sequence $\{x_n{}^2 + y_n{}^2\}$ converges to $0^2 + 0^2 = 0$. So all we need to show is that $\lim_{z\to 0^+} z\log z = 0$. This can be shown using L'Hopital's rule:

$$\lim_{z\to 0^+} z\log z = \lim_{t\to+\infty} \frac{1}{t}\log\frac{1}{t} = \lim_{t\to+\infty} -\frac{\log t}{t} = \lim_{t\to+\infty} -\frac{1/t}{1} = 0.$$

1.5.15: (a) $\lim_{\binom{x}{y}\to\binom{0}{0}} f\binom{x}{y} = a$ means that for any $\epsilon > 0$, there exists $\delta > 0$ such that whenever $\left|\binom{x}{y} - \binom{0}{0}\right| = \left|\binom{x}{y}\right| < \delta$ *and* $\binom{x}{y} \in D^*$, $\left|f\binom{x}{y} - a\right| < \epsilon$.

(b) First we show that the limit for f does not exist by approaching $\binom{0}{0}$ from two different lines $y = 0$ and $y = x$. If we approach $\binom{0}{0}$ from the line $y = 0$ along the positive x-axis, then the limit is $\lim_{x\to 0^+} f\binom{x}{0} = \lim_{x\to 0^+} \frac{\sin x}{x} = \lim_{x\to 0^+} \frac{\cos x}{1} = 1$, by using L'Hopital. If we approach $\binom{0}{0}$ from the line $y = x$ from the first quadrant, then the limit is $\lim_{x\to 0^+} f\binom{x}{x} = \lim_{x\to 0^+} \frac{\sin(2x)}{\sqrt{2}x} = \lim_{x\to 0^+} \frac{2\cos(2x)}{\sqrt{2}} = \sqrt{2}$, by using L'Hopital. Because we get two different values, the original limit must not exist. Second we show that the limit for g exists and is 0. It suffices to show that $\lim_{\binom{x}{y}\to\binom{0}{0}} |x| \log(x^2 + y^4)$ and $\lim_{\binom{x}{y}\to\binom{0}{0}} |y| \log(x^2 + y^4)$ exist and are both 0. Consider an arbitrary sequence $\{\binom{x_n}{y_n}\}$ that converges to $\binom{0}{0}$. Then ${x_n}^2 + {y_n}^4 \to 0$, so for sufficiently large n, ${x_n}^2 + {y_n}^4 < 1$ and $|x_n| \log({x_n}^2 + {y_n}^4) \le 0$. On the other hand, $|x_n| \log({x_n}^2 + {y_n}^4) \ge |x_n| \log({x_n}^2) = 2\,|x_n| \log(x_n)$. So for sufficiently large n,

$$2\,|x_n| \log(x_n) \le |x_n| \log({x_n}^2 + {y_n}^4) \le 0.$$

Now $x_n \to 0$ implies that $2\,|x_n| \log x_n \to 0$ (see 1.5.14(h)), so by a squeezing lemma, we know that $\lim_{\binom{x}{y}\to\binom{0}{0}} |x| \log(x^2 + y^4)$ must exist and must be 0. (You should try to formulate an appropriate squeezing lemma and prove it.) A similar argument (reversing the roles of x_n and y_n) shows that $\lim_{\binom{x}{y}\to\binom{0}{0}} |y| \log(x^2 + y^4)$ exists and is 0. Hence the original limit exists and is 0.

1.5.17: Suppose $a_{i(1)}, a_{i(2)}, \ldots$ is an infinite subsequence of $a_1, a_2, \ldots$. For a given $\epsilon > 0$, there exists N such that when $n > N$, $|a_n - a| < \epsilon$. Because $i(j) \to \infty$ as $j \to \infty$, there exists N' such that $i(N') > N$. (Alternatively, let $N' = N$. Then $i(N') \ge N' > N$.) For any $j > N'$, $i(j) > i(N') > N$ and hence $\left|a_{i(j)} - a\right| < \epsilon$. Thus by definition, $a_{i(j)} \to a$.

1.5.23: Part (a) is a special case of part (c), so it suffices to show part (c). Let f be the given map. Define maps $g, h\colon \mathbb{R}^2 \to \mathbb{R}$ by $g\begin{bmatrix} x \\ y \end{bmatrix} = ax + by$ and $h\begin{bmatrix} x \\ y \end{bmatrix} = cx + dy$. By corollary 1.5.28, g and h are continuous. The continuity of f follows from the following lemma, which follows directly from proposition 1.5.20 (why?):

6.1. Lemma. *Let $f_1, \ldots, f_n$ be real-valued functions defined on a domain $U \subset \mathbb{R}^n$, and define $f\colon U \to \mathbb{R}^n$ by $f(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{pmatrix}$. Then f is continuous at $a = \begin{pmatrix} a_1 \\ \vdots \\ )a_n \end{pmatrix}$ if and only if each f_i is continuous at a_i.*

1.5.24: The problem statement is vague in several senses, so we will just say that $\begin{bmatrix} 3.14 \\ 2.71 \end{bmatrix}$ and $\begin{bmatrix} 3.1415 \\ 2.7182 \end{bmatrix}$ are the first two that work, and that the a_n's thereafter also work. The latter is true because $\left| a_n - \begin{bmatrix} \pi \\ e \end{bmatrix} \right|$ decreases as n increases (why?).