MATH 23A SOLUTION SET 8

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- 2.2
  - (a)See top of p. 124 of Hubbard.
  - (b)Suppose f has an inverse on a neighborhood of 0. Without loss of generality, suppose f has an inverse on $(-\epsilon,\epsilon)$, for some $\epsilon>0$. Then f is monotone on $(-\epsilon,\epsilon)$ (Hubbard p. 219).
    For $x\neq 0$, $f'(x)=\frac{1}{2}-\cos\frac{1}{x}+2x\sin\frac{1}{x}$. For $x=\frac{1}{2\pi k}$ for any k, $f'(x)=\frac{1}{2}-1+2x\sin\frac{1}{x}$. We have $\lim_{x\to 0}2x\sin\frac{1}{x}=0$. So $\lim_{k\to\infty}f'(\frac{1}{2\pi k})=\lim_{x\to 0}(-\frac{1}{2}+2x\sin\frac{1}{x})=-\frac{1}{2}$. So we can find an integer k such that $\frac{1}{2\pi k}\in(0,\epsilon)$ and $f'(\frac{1}{2\pi k})<0$.
    From this negative derivative at a point, we will deduce that f is not increasing on $(-\epsilon,\epsilon)$. We have
    $$\lim_{h\to 0}\frac{f(\frac{1}{2\pi k}+h)-f(\frac{1}{2\pi k})}{h}<0.$$
    So there exists $\delta>0$ such that for all $|h|<\delta$,
    $$\frac{f(\frac{1}{2\pi k}+h)-f(\frac{1}{2\pi k})}{h}<0.$$
    Let $h=\min(\frac{\delta}{2},\frac{1}{2}(\epsilon-\frac{1}{2\pi k}))$. Then $\frac{1}{2\pi k}+h<\epsilon$. Since $h>0$ and $|h|<\delta$, $f(\frac{1}{2\pi k}+h)-f(\frac{1}{2\pi k})<0$. Since $\frac{1}{2\pi k},\frac{1}{2\pi k}+h\in(-\epsilon,\epsilon)$ and $\frac{1}{2\pi k}<\frac{1}{2\pi k}+h$ but $f(\frac{1}{2\pi k})>f(\frac{1}{2\pi k}+h)$, f is not increasing on $(-\epsilon,\epsilon)$.
    Also, for $x=\frac{1}{(2m+1)\pi}$, $f'(x)=\frac{1}{2}+1+2x\sin\frac{1}{x}$. So $\lim_{m\to\infty}f'(\frac{1}{(2m+1)\pi})=\lim_{x\to 0}(\frac{3}{2}+2x\sin\frac{1}{x})=\frac{3}{2}$. So we can find an integer m such that $\frac{1}{(2m+1)\pi}\in(0,\epsilon)$ and $f'(\frac{1}{(2m+1)\pi})>0$.
    Similarly, we have
    $$\lim_{h\to 0}\frac{f(\frac{1}{(2m+1)\pi}+h)-f(\frac{1}{(2m+1)\pi})}{h}>0.$$
    So there exists $\delta_2>0$ such that for all $|h|<\delta_2$,
    $$\frac{f(\frac{1}{(2m+1)\pi}+h)-f(\frac{1}{(2m+1)\pi})}{h}<0.$$
    Let $h_2=\min(\frac{\delta}{2},\frac{1}{2}(\epsilon-\frac{1}{(2m+1)\pi}))$. Then $\frac{1}{(2m+1)\pi}+h_2<\epsilon$. Since $h_2>0$ and $|h|<\delta_2$, $f(\frac{1}{(2m+1)\pi}+h)-f(\frac{1}{(2m+1)\pi})>0$. Since $\frac{1}{(2m+1)\pi},\frac{1}{(2m+1)\pi}+h\in(-\epsilon,\epsilon)$ and $\frac{1}{(2m+1)\pi}<\frac{1}{(2m+1)\pi}+h\in(-\epsilon,\epsilon)$ but $f(\frac{1}{(2m+1)\pi})<f(\frac{1}{(2m+1)\pi}+h)$, f is not decreasing on $(-\epsilon,\epsilon)$. So f is not monotone on $(-\epsilon,\epsilon)$. This is a contradiction.
    So we can conclude from this contradiction that our hypothesis that f has an inverse on a neighborhood of 0 is false. So f does not have an inverse on any neighborhood of 0.

- (c)This does not contradict Theorem 2.9.2 because f is not monotone on any interval around 0. More generally, our result does not contradict the Inverse Function Theorem because f' is not continuous at 0. This is because the limits as k and m approach ∞ tell us that in every interval around 0 we can find a point x_1 such that $f(x_1)$ is as close to $-\frac{1}{2}$ as we wish and a point x_2 such that $f(x_2)$ is as close to $\frac{3}{2}$ as we wish.

• 2.3

- (a)Let $f(x, y) = y^2 + y + 3x + 1$. We can use the Quadratic Formula to find all (x,y) satisfying $f(x, y) = 0$:

$$\{(x, y) : y = \frac{-1 \pm \sqrt{1 - 4(3x + 1)}}{2}\},$$

which is equivalent to

$$\{(x, y) : y = y_1(x) \text{ or } y = y_2(x)\}$$

where

$$y_1(x) = \frac{-1 + \sqrt{1 - 4(3x + 1)}}{2}$$

and

$$y_2(x) = \frac{-1 - \sqrt{1 - 4(3x + 1)}}{2}.$$

This gives us two choices for y as a function of x, each defined where the discriminant in the above equation is nonnegative: $x \leq -\frac{1}{4}$. Where $y_1(x) = y_2(x)$, y is not defined implicitly as a function of x. This is because, using continuity of y_1 and y_2, we can find, for any ϵ-ball around (x, y), sufficiently small k such that $(x+k, y_1(x+k))$ and $(x+k, y_2(x+k))$ are both in the ϵ-ball. The intersection occurs when

$$\frac{-1 + \sqrt{1 - 4(3x + 1)}}{2} = \frac{-1 - \sqrt{1 - 4(3x + 1)}}{2},$$

which occurs only when $x = -\frac{1}{4}$ and $y = -\frac{1}{2}$.

Suppose $(x, y) \neq (-\frac{1}{4}, -\frac{1}{2})$ lies on one of these two curves (WLOG, y_1). In the neighborhood of x $(x - \epsilon, x + \epsilon)$, where $\epsilon = |(x - \frac{1}{4})|$, y_1 and y_2 do not intersect and are monotonous. So in this neighborhood of x and in the neighborhood of y $(y_1(x - \epsilon), y_1(x + \epsilon))$, y is defined implicitly as a function of x. So y is locally defined implicitly as a function of x on

$$\{(x, y) : x < -\frac{1}{4}, y = \frac{-1 \pm \sqrt{1 - 4(3x + 1)}}{2}\}.$$

- (b)The implicit function theorem for $f\colon \mathbb{R}^2 \to \mathbb{R}$ states that if $f(x, y) = 0$ and $\mathrm{D}f_2(x, y) \neq 0$, then y is implicitly defined as a function of x. The quadratic formula (in part (a)) tells us that $f(x, y) \neq 0$ whenever $x > -\frac{1}{4}$ and finds y in terms of x whenever $x \leq -\frac{1}{4}$. Also, $\mathrm{D}\, f_2(x, y) = 2y + 1$, which is nonzero whenever $y \neq -\frac{1}{2}$ and zero when $y = -\frac{1}{2}$. So $\mathrm{D}f_2(x, y)$ is nonzero whenever $x \neq -\frac{1}{4}$ and zero when $x = -\frac{1}{4}$. So the implicit function theorem tells us that y is defined implicitly as a function of x when $x < -\frac{1}{4}$. And as above we see that y is not defined implicitly as a function of x in any ball around $(-\frac{1}{4}, -\frac{1}{2})$.

- Lemma 1: The mapping $S\colon \text{Mat}(2,2) \to \text{Mat}(2,2)$ given by $S(A) = A^2$ is continuously differentiable for all A.

  Proof: Consider a 2-by-2 matrix as an element of $\mathbb{R}^4$ through the following correspondence:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longleftrightarrow \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

  Then for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $H = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$, we have

$$AH + HA = \begin{bmatrix} 2ae + bg + cf & af + be + bh + df \\ ag + ce + dg + ch & cf + bg + 2dh \end{bmatrix} \longleftrightarrow \begin{bmatrix} 2ae + bg + cf \\ af + be + bh + df \\ ag + ce + dg + ch \\ cf + bg + 2dh \end{bmatrix}$$

  So the Jacobian matrix for $[\mathbf{D}S(A)]$ is

$$\begin{bmatrix} 2a & c & b & 0 \\ b & a+d & 0 & b \\ c & 0 & a+d & c \\ 0 & c & b & 2d \end{bmatrix}$$

  and we can see that all partial derivatives of S are continuous, so S is continuously differentiable.
  - 2.8

    (a)Yes. To apply the Inverse Function Theorem to achieve our desired result, it suffices to show that S is continuously differentiable in an open set containing $-I$ and that $[\mathbf{D}S(-I)]$ is invertible. We have shown the former in Lemma 1. We have seen before that $[\mathbf{D}S(A)]H = AH + HA$. So $[\mathbf{D}S(-I)]H = (-I)H + H(-I) = -H - H = -2H$. So $[\mathbf{D}S(-I)]$ is clearly invertible, with inverse defined by $[\mathbf{D}S(-I)]^{-1}(H) = -\frac{1}{2}H$. So the Inverse Function Theorem tells us that there exists an inverse mapping g such that $S(g(A)) = A$, defined in a neighborhood of I, such that $g(I) = -I$.
- 2.9 False. We denote the squaring map by S. One way to solve both this problem and 2.12 below is to compute $\mathbf{D}S\left(\begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}\right)$ and see that it is not invertible and its determinant is zero. Then let $r > 0$ be arbitrary and suppose that there exists a differentiable map $g\colon B_r\left(\begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}\right) \to \text{Mat}(2,2)$ such that $g\left(\begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}\right) = \left(\begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}\right)$ and $S(g(A)) = A$ for all $A \in B_r\left(\begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}\right)$. By the Chain Rule, $\mathbf{D}(S \circ g)\left(\begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}\right) = \mathbf{D}S\left(\begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}\right)\mathbf{D}g\left(\begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}\right)$. But $(S \circ g)$ is the identity map, so $\mathbf{D}(S \circ g)\left(\begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}\right) = I$. So $1 = \det I = \left(\det \mathbf{D}S\left(\begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}\right)\right)\left(\det \mathbf{D}g(\left(\begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}\right)\right)$. So $1 = 0$. So we have a contradiction and therefore such a map g does not exist.

Here's a solution that I wrote up before I realized this (it actually proves a stronger result):

Let $r > 0$ be arbitrary and suppose that for all $A \in B_r\left(\begin{bmatrix}-3 & 0\\ 0 & -3\end{bmatrix}\right)$ there exists g(A) such that $g(A)^2 = A$. Choose A= $\begin{bmatrix}-3 & 0\\ 0 & -3+\frac{r}{2}\end{bmatrix}$. Then we have $g(A) = \begin{bmatrix}a\ b\\ c\ d\end{bmatrix}$ such that $g(A)^2 = \begin{bmatrix}a^2+bc & ab+bd\\ ac+cd & bc+d^2\end{bmatrix} = \begin{bmatrix}-3 & 0\\ 0 & -3+\frac{r}{2}\end{bmatrix}$. So $b(a+d) = 0 \Rightarrow b = 0$ or $a + d = 0$. If $b = 0$, then $a^2 = -3$ and we get a contradiction. So $a + d = 0$ and $d = -a$. But examining the upper left and lower right entries of $g(A)^2$ tells us that $d^2 - a^2 = \frac{r}{2}$. But $d = -a \Rightarrow d^2 = a^2$. We have a contradiction. So there is no mapping g such that $g(A)^2 = A$ for the particular A we have chosen and definitely no mapping g such that $g(A)^2 = A$ for all $A \in B_r\left(\begin{bmatrix}-3 & 0\\ 0 & -3\end{bmatrix}\right)$.

- 2.11
  - (a) To apply the Inverse Function Theorem to show that $F\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}e^x+e^y\\ e^x+e^{-y}\end{pmatrix}$ is locally invertible at $\begin{pmatrix}x\\ y\end{pmatrix}$, it suffices to show that F is continuously differentiable at $\begin{pmatrix}x\\ y\end{pmatrix}$ and that D$F\begin{pmatrix}x\\ y\end{pmatrix}$ is invertible. We see that all the partial derivatives of F are continuously differentiable and use them to write the Jacobian D$F\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}e^x & e^y\\ e^x & -e^{-y}\end{pmatrix}$. Since $e^x > 0$ for all x, we know that for all x and y the first column contains two positive entries and the second column a positive entry and a negative entry. So neither column is a linear multiple of the other, so the columns are linearly independent. So D$F\begin{pmatrix}x\\ y\end{pmatrix}$ is invertible. So the Inverse Function Theorem tells us that F is locally invertible at every point $\begin{pmatrix}x\\ y\end{pmatrix} \in \mathbb{R}^2$.
  - (b)The Inverse Function Theorem tells us that $(\mathrm{D}F^{-1})(b) = (\mathrm{D}F(b))^{-1} = \frac{1}{e^{x-y}+e^{x+y}}\begin{bmatrix}e^{-y} & e^y\\ e^x & -e^x\end{bmatrix}$ by 1.2.12 from Problem Set 1 or your favorite technique to invert a 2x2 linear map.
- Lemma 2: Let $F\colon U \to V$ be differentiable at $a \in U$. Then F is continuous at a.

  Proof: There is a linear transformation L such that

$$\lim_{|h|\to 0} \frac{(F(a+h) - F(a)) - L(h)}{|h|} = 0.$$

  So $\forall \epsilon > 0 \exists \delta > 0$ such that $|h| < \delta$

$$\Rightarrow \frac{|(F(a+h) - F(a)) - L(h)|}{|h|} < \epsilon$$

$$\Rightarrow |F(a+h) - F(a) - L(h)| < \epsilon|h|$$

$$\Rightarrow |F(a+h) - F(a)| - |L(h)| < \epsilon|h|$$

$$\Rightarrow |F(a+h) - F(a)| < \epsilon|h| + L(h) \leq \epsilon|h| + |L||h|$$
$$\Rightarrow \lim_{h\to 0} |F(a+h) - F(a)| = 0$$
$$\Rightarrow \lim_{h\to 0} F(a+h) = F(a)$$
$$\Rightarrow F \text{ is continuous at } a$$

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- 2.12 False. Suppose there is a neighborhood $U \subset \mathrm{Mat}(2,2)$ and a C^1 mapping $F : U \to \mathrm{Mat}(2,2)$ such that $F\left(\begin{bmatrix}5 & 0\\0 & 5\end{bmatrix}\right) = \begin{bmatrix}1 & 2\\2 & -1\end{bmatrix}$ and $S(F(A)) = A$, where S denotes the squaring map. Since F is continuously differentiable, by the Inverse Function Theorem there exists an open V containing $\begin{bmatrix}5 & 0\\0 & 5\end{bmatrix}$ and an open W containing $\begin{bmatrix}1 & 2\\2 & -1\end{bmatrix}$ and a unique differentiable inverse (to F) $T : W \to V$ such that $\mathrm{D}T\left(\begin{bmatrix}1 & 2\\2 & -1\end{bmatrix}\right) = \left(\mathrm{D}F\left(T\left(\begin{bmatrix}1 & 2\\2 & -1\end{bmatrix}\right)\right)\right)^{-1}$. But since the inverse T is unique, T=S. So $\left(\mathrm{D}S\left(\begin{bmatrix}1 & 2\\2 & -1\end{bmatrix}\right)\right)^{-1} = \mathrm{D}F\left(T\left(\begin{bmatrix}1 & 2\\2 & -1\end{bmatrix}\right)\right)$. But we can check that $\mathrm{D}S\left(\begin{bmatrix}1 & 2\\2 & -1\end{bmatrix}\right)$ is not invertible. So we have a contradiction. So such a F does not exist.

  Another way to do the problem: Suppose that there exists $r > 0$ such that for $U = B_r\left(\begin{bmatrix}5 & 0\\0 & 5\end{bmatrix}\right)$ there exists a continuously differentiable mapping $F\colon U \to \mathrm{Mat}(2,2)$ such that $F\left(\begin{bmatrix}5 & 0\\0 & 5\end{bmatrix}\right) = \begin{bmatrix}1 & 2\\2 & -1\end{bmatrix}$ and $F(A)^2 = A$ for all $A \in U$. Since F is differentiable, F must be continuous by Lemma 2. So there exists $\delta > 0$ such that $\left|A - \begin{bmatrix}5 & 0\\0 & 5\end{bmatrix}\right| < \delta \Rightarrow \left|F(A) - \begin{bmatrix}1 & 2\\2 & -1\end{bmatrix}\right| < 1$. Choose $k < \min(\frac{r}{2}, \delta)$ and $A = \begin{bmatrix}5 & 0\\0 & 5+k\end{bmatrix}$. So $\left|A - \begin{bmatrix}5 & 0\\0 & 5\end{bmatrix}\right| < \delta$. Let us write $F(A) = \begin{bmatrix}a & b\\c & d\end{bmatrix}$. Then $F(A)^2 = \begin{bmatrix}a^2+bc & ab+bd\\ac+cd & bc+d\end{bmatrix} = \begin{bmatrix}5 & 0\\0 & 5+\frac{r}{2}\end{bmatrix}$. So $b(a+d) = 0 \Rightarrow b = 0$ or $a + d = 0$. If $b = 0$, then $a^2 = 5$ so $|a-1| > 1$ so $\left|F(A) - \begin{bmatrix}1 & 2\\2 & -1\end{bmatrix}\right| > 1$, which contradicts our earlier result from the continuity of F.

  So $a + d = 0$ and $d = -a$. But examining the upper left and lower right entries of $g(A)^2$ tells us that $d^2 - a^2 = k$. But $d = -a \Rightarrow d^2 = a^2$. We have a contradiction. So there is no mapping F satisfying the given conditions.