MATH 23A
SOLUTION SET 9

NILS R. BARTH
December 11, 1999

1: (a) First, $f(2,2)=0$, so $(2,2)\in C_0$. Second,

$$[Df](x,y) = \begin{bmatrix} 3x^2+6x-12 & 2y-6 \end{bmatrix}, \tag{9.1}$$

so $Df(2,2) = [\,12\ \ {-2}\,]$, which is not equal to zero, so $(2,2)$ is a smooth point.

(b) Since $D_2f(2,2)\neq 0, D_1f(2,2)\neq 0$, we conclude from the inverse function theorem that (respectively), there exist ϵ_f,ϵ_g such that $y=f(x), x=g(y)$ on $(2-\epsilon_f, 2+\epsilon_f$ and $(2-\epsilon_g, 2+\epsilon_g)$, respectively. Taking $\epsilon=\min(\epsilon_f,\epsilon_g)$ yields the desired result.

(c) The tangent line of the curve $f(x,y)=0$ at (x_0,y_0) is given by $[D_1f](x_0,y_0)\cdot(x-x_0)+[D_2f](x_0,y_0)\cdot(y-y_0)=0$. In our case, this yields $12(x-2)-2(y-2)=0$.
The tangent line is a kind of an affine copy of the tangent space, which is given by $[D_1f](x_0,y_0)\dot{x}+[D_2f](x_0,y_0)\dot{y}=0$, or in our case $12\dot{x}-2\dot{y}=0$.

(d) It is natural to solve this and the next part simultaneously. C_c will be singular at *some* point iff $Df=[\,0\ \ 0\,]$ at some point of C_c. Now $Df=0$ for $(x,y)=(-1\pm\sqrt{5},3)$ (by solving the linear and quadratic polynomials in (9.1)), so C_c is singular iff $(-1\pm\sqrt{5},3)\in C_c$, i.e., iff $c=f(-1\pm\sqrt{5},3)$.

(e) See above; the singular points of $C_{f(-1\pm\sqrt{5},3)}$ are $(-1\pm\sqrt{5},3)$, respectively.

(f) By reasoning as above, $Df(x,y)=[\,p'(x)\ \ {-q'(y)}\,]$. This is zero iff $p'(x)=q'(y)=0$. Now p',q' are polynomials of degrees $m-1, n-1$, respectively, so they have roots (possibly repeated) $x_1,\dots,x_{m-1},y_1,\dots,y_{n-1}$. So (x,y) is a singular point iff $Df(x,y)=0$, that is, for (x_i,y_j). Now these points lie on the curves $C_{f(x_i,y_j)}$, and thus there are at most $(m-1)(n-1)<mn$ singular curves.

2: (a) Note that squaring is locally C^1 invertible at a point iff the Jacobian at that point is invertible (and the partials are C^1, which they are). Via the attached Mathematica output, we see that

$$DS\begin{bmatrix} a & 0\\ 0 & b\end{bmatrix} = \begin{bmatrix} 2a & & & \\ & a+b & & \\ & & a+b & \\ & & & 2b\end{bmatrix}. \tag{9.2}$$

Further, note that a diagonal matrix is invertible iff all its diagonal entries are non-zero, and we see that the desired g exists iff $2a\neq 0$, $2b\neq 0$, and $a+b\neq 0$. A slick way of saying all of this is: $ab(a+b)\neq 0$.

(b) As above, but now we get

$$DS\begin{bmatrix} a & 0\\ 0 & b\end{bmatrix} = \begin{bmatrix} 5a^4 & & & \\ & a^4+a^3b+a^2b^2+ab^3+b^4 & & \\ & & a^4+a^3b+a^2b^2+ab^3+b^4 & \\ & & & 5b^4\end{bmatrix}, \tag{9.3}$$

so g exists iff $ab(a^4+a^3b+a^2b^2+ab^3+b^4)\neq 0$.
Incidentally, you may have noticed that the big middle term is also $\frac{a^5-b^5}{a-b}$, so the middle term factors as $\prod_{i=1}^4(a-\zeta_5^i b)$, where $\zeta=e^{2\pi i/5}$ (we call ζ_5 a *primitive fifth root of unity*). This is because $a^n-b^n=\prod_{i=1}^n(a-\zeta_n^i b)$, and we've just gotten rid of the $(a-b)$ term.
Looking at this, one could conjecture that for diagonal matrices, the map $A\mapsto A^n$ is locally C^1 invertible iff $ab\frac{a^n-b^n}{a-b}\neq 0$ (where one actually divides out the fraction). Can you prove this? Hint: it follows almost immediately from the expansion $Df(A)(H)=HA^{n-1}+AHA^{n-2}+\cdots+A^{n-1}H$ for $f(A)=A^n$.