

## MATH 23A PROBLEM SET 11

Let  $V$  be a finite-dimensional vector space over  $\mathbb{R}$ . We denote by  $C^\infty(V)$  the space of functions  $f$  on  $V$  such that  $f$  has continuous derivatives of all orders. In particular we denote by  $C^\infty(\mathbb{R})$  the space of functions  $f$  on  $\mathbb{R}$  such that  $f$  has continuous derivatives of all orders. If  $f \in C^\infty(\mathbb{R})$ ,  $a \in \mathbb{R}$  we define the Taylor series  $S_f(a, x)$  for  $f$  at  $a$  by  $S_f(a, x) = \sum_{n \geq 0} f^{(n)}(a)/n!(x - a)^n$ .

1. Prove that for any  $a, x \in \mathbb{R}$  the Taylor series series  $S_f(a, x)$  is convergent to  $f(x)$  for  $f(x) = \exp(x)$ .

2. Let  $f$  be a function on  $\mathbb{R}$  such that  $f(x) = 0$  if  $x \leq 0$  and  $f(x) = \exp(-1/x)$  for  $x > 0$ .

a) Show that  $f \in C^\infty(\mathbb{R})$ .

b) Find the Taylor series  $S_f(0, x)$ .

c) Explain why does not  $S_f(0, x)$  converge to  $f(x)$ .

3. Let  $V$  be a finite-dimensional vector space over  $\mathbb{R}$ ,  $f \in C^\infty(V)$ ,  $v \in V$ .

a) Define the linear Taylor polynomial  $P_1(v, x)$ ,  $x \in V$  for  $f$ .

b) Define the quadratic Taylor polynomial  $P_2(v, x)$ ,  $x \in V$  for  $f$ .

c) Define the Taylor polynomial  $P_n(v, x)$ ,  $x \in V$  for  $f$ .