MATH 23A
LECTURE NOTES

NILS R. BARTH
November 5, 1999

- The derivative as tangent plane $\mathbf{R}^n \to \mathbf{R}$ *draw picture*.
- Algebraically, $f(x+h) = f(x) + [L(x)](h) + o(|h|)$; for $n = 1$ this is $f(x+h) = f(x) + [f'(x)] * h + o(h)$.
- Partial derivatives: $f = (f_1, \dots, f_m)\colon \mathbf{R}^n \to \mathbf{R}^m$

$$D_i f_j(v) = \frac{\partial f_j}{\partial x_i}(v) = \lim_{h \to 0} \frac{f_j(v + he_i) - f_j(v)}{h} \tag{19991105.1}$$

- Partial derivatives as axes of tangent plane, so they determine the (total) derivative (if it exists).
- Computation rule: when taking the partial derivative with respect to x_i, just hold other variables constant, e.g., $D_x(\sin x \tan y) = \cos x \tan y$; after all, the other variables *are* constant.
- Partials summarized in Jacobian:

$$f'(x) = \begin{pmatrix} D_1 f_1(x) & \dots & D_n f_1(x) \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ D_1 f_m(x) & \dots & D_n f_m(x) \end{pmatrix} \tag{19991105.2}$$

  To remember which way indices go, think $m = 1$; then it's a covector.
- **19991105.3. Theorem** (1.7.10). *If the derivative exists, then it is the Jacobian; in particular, partial derivatives exist.*

  Proof. Apply definition of derivative on basis vectors. □
- If derivative exists, then directional derivative given by linear combination of axes, i.e., $D_u f(x) = [f'(x)](u)$
- Warning: all partial derivatives do not guarantee existence of derivative; consider the surface swept out by a line through the origin in $\mathbf{R}^3$ moving with varying slope (i.e., a map $\mathbf{R}^2 \to \mathbf{R}$) that vanishes on the axes.
- Another example: $\frac{x^2 y}{x^2+y^2}$ (check that $\lim_{x,y\to 0} f(x,y) = 0$). Then $D_1 f(0) = D_2 f(0) = 0$, so $f'(0) = 0$, but $D_{(1,1)} f = \frac{1}{2}$.
- This is the general case; in some sense, virtually every function is ill-behaved.
- If partial derivatives exist and are continuous in neighborhood of x, then f is differentiable at x (check that above example has non-continuous partials; look at limits along $x = y$ and $y = 0$).
- Sketch of proof (proof p. 125–6): Only varies linearly in each dimension, so overall varies by n times linearly, which is still linearly.