MATH 23A FINAL

1. (10 pts) Let V be a finite-dimensional vector space,$V', V'' \subset V$ linear subspaces such that $V' \cap V'' = \{0\}$ and $dim(V) = dim(V') + dim(V'')$. Prove that for any $v \in V$ there exist $v' \in' V$ and $v'' \in V''$ such that $v = v' + v''$. Show also that for any $v \in V$ such a pair (v', v'') is unique.

2.a) (10 pts) Let $X \subset \mathbb{R}$ be a non-empty subset which is similtoneously open and closed. Show that $X = \mathbb{R}$.

[A hint. Since X is non empty you can fix a point $x_0 \in X$. Look for the "smallest" point $y \in \mathbb{R} - X$ which is bigger then x_0 and show that an existence of y would lead to a contradiction]

b) (5 pts) Let $X \subset \mathbb{R}^n$ be a non-empty subset which is similtoneously open and closed. Show that $X = \mathbb{R}^n$.

3)a) (10 pts) Let V be a vector space, $A : V \to V$ a linear operator $v_1, v_2 \in V$ non-zero vectors such that $Av_1 = c_1v_1, Av_2 = c_2v_2$ where $c_1 \neq c_2 \in \mathbb{R}$. Prove that the vectors v_1, v_2 are linearly independent.

b) (10 pts) Let $v_1, \dots, v_n$ be non-zero vectors in V such that $Av_1 = c_1v_1, \dots,\ Av_n = c_nv_n$ where $c_1, \dots, c_n \in \mathbb{R}$ are such that $c_i \neq c_j$ for $i \neq j$. Prove that the vectors $v_1, \dots, v_n$ are linearly independent.

4) (5 pts) Let $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$ be a twice differentiable function with continuous second derivatives,$g(h) = f(h, h^3)$. Prove that $g''(0) = f_{xx}(0, 0)$.

5)Let $f(x, y, z) : \mathbb{R}^3 \to \mathbb{R}$ be a continuously differentiable function such that $f(0, 0, 0) = 0$ and $f_z(0, 0, 0) \neq 0$.

a) (5 pts) Show that there exists an open neighboorhood $U \subset \mathbb{R}^2$ of the point $(0, 0)$ and a continuously differentiable function $g(x, y)$ on U such that $f(x, y, g(x, y)) = 0$.

b) (5 pts) Assume that $g(0, 0) \leq g(x, y)$ for all $(x, y) \in U$. Show that $f_x(0, 0, 0) = f_y(0, 0, 0) = 0$.

6)Let $P_n(x)$ be the Taylor polynomial of degree n for $f(x) = \sin(x)$ at $x_0 = 1$.

a) (5 pts) Find $P_3(x)$.

b) (10 pts) Let $r_n(x) = \sin(x) - P_n(x)$. Prove that $\lim_{n\to\infty} r_n(10) = 0$