THE GRAMIAN AND k-VOLUME IN n-SPACE: SOME CLASSICAL RESULTS IN LINEAR ALGEBRA

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ABSTRACT. We give a formula for determining when a set of k vectors in n-space is linearly independent, and if so, what is the volume of the parallelepiped with these vectors as its sides. This function, the gramian, allows one to partially apply the determinant when the number of vectors you have is less than the dimension of the ambient space. These results were classically known, but are not part of the standard linear algebra curriculum. Prerequisites: familiarity with matrices and determinants.

CONTENTS

1. INTRODUCTION

Given k vectors in n-dimensional space, when are they linearly independent, and what is the k-volume of the parallelepiped that they define? When $k = n$, the answer is familiar: arrange the vectors in a matrix and take the determinant. The vectors are then linearly dependent iff the determinant is zero, and otherwise the volume of the parallelepiped, $\mathrm{Vol}(v_1, \ldots, v_k)$, is the absolute value of the determinant. For $k < n$, the widely known answers are piecemeal: for $k = 2, n = 3$, one can use the cross product; to determine linear dependence, one can use gaussian elimination, though this is an algorithm, rather than a formula. Ideally, one would have some analog for the determinant which applies when the number of vectors is not equal to the dimension of the ambient space. This answer is provided in the gramian, which we introduce below; we loosely follow [1, §IX.3–5, p. 246–56].

2. NOTATION AND REVIEW

Let us fix a vector space K^n, where K is a subfield of the complex numbers. For example, let $K = \mathbf{R}$, the real numbers, which is euclidean space; the other main example is the complex numbers, and there is little loss of generality in only considering these. We write vectors vertically and covectors (elements of the dual

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space $(K^n)^*$) horizontally. We write $W < V$ to indicate that W is a subspace of V. Given a collection of vectors $\{v_1, \ldots, v_k\}$, denote the parallelepiped with sides $v_1, \ldots, v_k$ by $P(v_1, \ldots, v_k)$.

If $\{v_1, \ldots, v_k\}$ is linearly dependent, this definition doesn't make sense; more formally,

$$P(v_1, \ldots, v_k) = \{\textstyle\sum a_i v_i \mid 0 \leq a_i \leq 1, \sum a_i \leq 1\}. \tag{1}$$

When $\{v_1, \ldots, v_k\}$ are linearly independent, $P(v_1, \ldots, v_k)$ has a well-defined k-volume (volume as a subset of k-dimensional space); when they are linearly dependent, this volume is zero. Note that if you have, for instance, 3 vectors in 2-space, then $P(v_1, \ldots, v_3)$ has non-zero volume as a subset of 2-space, but it is "flat" and thus has zero volume as a subset of 3-space.

Recall that an *inner product* is a choice of distance (and also volume) in a vector space; the main example is the *dot product*, defined as

$$v \cdot w = v^* w, \tag{2}$$

which is a scalar, and where $v^* = \overline{v}^T$ is the complex conjugate of the transpose. We take complex conjugates so that complex vectors have real length; if $K = \mathbf{R}$, this reduces to $v \cdot w = v^T w$. We define the *norm* (length) of v as

$$\|v\| = v \cdot v = v^* v. \tag{3}$$

Matrices that preserves distances and angles, i.e., $Ov \cdot Ow = v \cdot w$, are of particular interest, and are called *orthogonal*;[1] these correspond to rigid motions. For instance, orthogonal transforms of $\mathbf{R}^2$ are rotations and reflections.

3. The gramian and its basic properties

Definition 4 (gramian)**.** *The* gramian *of* $v_1, \ldots, v_k$ *is* $G(v_1, \ldots, v_k) = \det M^* M$, *where*

$$M = \begin{pmatrix} v_1 & \cdots & v_k \end{pmatrix} \tag{5}$$

and $M^* = \overline{M}^T$ *is the conjugate transpose of* M.

Note the similarity to the norm (3); in fact, $G(v) = \|v\|$, so the gramian of a single vector is the length, i.e., the 1-dimensional volume, of that vector.

If we calculate the entries of $M^* M$, we obtain the following characterization of the gramian:

Definition 6 (gramian—alternate)**.** $G(v_1, \ldots, v_k) = \det A$, *where* $a_{ij} = v_i \cdot v_j$.

This presentation shows that the matrix $M^* M$ captures all the geometric information about the vectors $\{v_1, \ldots, v_k\}$: the length of the vectors and the angles between them. Further, this is the only information that it contains; we've lost the particular orientations of the vectors and their embedding in n-space. We claim that this is enough information to easily determine the volume.

Consider the case where $k = n$; then $\det M^* M = \overline{(\det M)}(\det M) = |\det M|^2$. In particular, $G(v_1, \ldots, v_k) \geq 0$. Since $\operatorname{Vol}(v_1, \ldots, v_k) = |\det M|$, we obtain:

$$\operatorname{Vol}(v_1, \ldots, v_k) = \sqrt{\det M^* M} = \sqrt{G(v_1, \ldots, v_k)}. \tag{7}$$

[1] If the vector space is complex, they are instead called *unitary*.

The (i,j)-th entry of $A = M^*M$ is $v_i \cdot v_j$; the above shows that the volume depends only on these inner products.

For any collection of k vectors, the above holds by simply restricting to a k-dimensional subspace containing them; that is, $\mathrm{Vol}(v_1, \ldots, v_k) = \det A$. By using (6), we obtain the following geometric characterization of the gramian:

Proposition 8. $\mathrm{Vol}(v_1, \ldots, v_k) = \sqrt{G(v_1, \ldots, v_k)}$.

Note that in case $k \neq n$, we still have $G(v_1, \ldots, v_k) \geq 0$, so this is well-defined.

Proof. The proof was sketched above; we fill in the details here.

Given $v_1, \ldots, v_k$, where $k \leq n$, assume that they are linearly independent[2] and pick an orthonormal basis $w_1, \ldots, w_k$ for their span, W. Extend to an orthonormal basis $w_1, \ldots, w_n$ for K^n; the matrix O sending $w_i \mapsto e_i$ is an orthogonal change of coordinates, so it does not change inner products: $O(v) \cdot O(w) = v \cdot w$. Let $v_i' = O(v_i)$; then

$$(v_i') = \begin{bmatrix} m_{1i} \\ m_{2i} \\ \vdots \\ m_{ki} \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

since v_i is in the span of $w_1, \ldots, w_k$. The set $\{v_1', \ldots, v_k'\}$ visibly lies in the k-dimensional subspace of vectors with last $n-k$ coordinates zero, so the k-volume of $P(v_1', \ldots, v_k')$ is $|\det M|$, where m_{ij} as above. Effectively, we are restricting to the subspace W. By the discussion before this proposition, $\mathrm{Vol}(v_1', \ldots, v_k') = \sqrt{\det M^*M}$ (we're just dropping the trailing zeros). Since O is orthogonal,

$$\mathrm{Vol}(v_1', \ldots, v_k') = \mathrm{Vol}(v_1, \ldots, v_k) \tag{9}$$

and $v_i' \cdot v_j' = v_i \cdot v_j$, so

$$\sqrt{G(v_1, \ldots, v_k)} = \mathrm{Vol}(v_1', \ldots, v_k') = \mathrm{Vol}(v_1, \ldots, v_k), \tag{10}$$

as desired.

When $k > n$, the vectors are linearly dependent, so the k-volume of $P(v_1, \ldots, v_k)$ is zero. In this case the gramian is zero, by the argument in the proof of (11), below. ☐

In particular, the k-volume of $P(v_1, \ldots, v_k)$ is zero iff $\{v_1, \ldots, v_k\}$ don't form the sides of a k-parallelepiped; that is, if $P(v_1, \ldots, v_k)$ has dimension less than k. This yields:

Corollary 11 (Gram's criterion for linear dependence of vectors)**.** *The set of vectors $\{v_1, \ldots, v_k\}$ is linearly dependent iff $G(v_1, \ldots, v_k) = 0$.*

Proof. We just dealt with the case $k \leq n$; it remains to show that if $k > n$, the gramian is zero. Consider $A = M^*M$; each row of the $k \times k$ matrix A is a linear combination of rows of M, of which there are n. Thus, the row-rank of A is at most $n < k$, so its determinant is zero. ☐

[2] If they are not linearly independent, instead take W to be some k-dimensional subspace containing them.

Note that the gramian is always nonnegative,[3] and equals zero iff the vectors are linearly dependent. As a consequence, we obtain the familiar:

Corollary 12 (Bunyakovskiĭ-Cauchy-Schwarz inequality)**.** *$(v \cdot w)^2 \leq \|v\| \, \|w\|$, with equality iff v, w are linearly dependent (one is a multiple of the other).*

Proof. $G(v, w) \geq 0$, with equality iff v, w are linearly dependent. Now

$$G(v,w) = \det \begin{pmatrix} v \cdot v & v \cdot w \\ w \cdot v & w \cdot w \end{pmatrix} = \det \begin{pmatrix} \|v\| & v \cdot w \\ v \cdot w & \|w\| \end{pmatrix} = \|v\| \, \|w\| - (v \cdot w)^2; \tag{13}$$

the result follows. □

4. Extensions

The above discussion with matrices didn't depend on choice of coordinates, so it holds for any linear transform $T \colon V \to W$, so long as you have fixed inner products on V, W, which is equivalent to isomorphisms $V \xrightarrow{\sim} V^*, W \xrightarrow{\sim} W^*$. This yields

$$V \xrightarrow{\ T\ } W \xrightarrow{\ \sim\ } W^* \xrightarrow{\ T^*\ } V^* \xrightarrow{\ \sim\ } V, \tag{14}$$

the composition of which is an endomorphism of V (linear operator on V), and thus has a determinant. The gramian of T (with respect to the inner products on V, W) is then the determinant of this map.

Recall that the inner product of v, w with respect to an inner product other than the dot product is given by $\overline{v}^T P w$, where P is some invertible matrix. Similarly, $M^* P M$ is the gramian with respect to this inner product, which follows from expanding (6).

The key part of the restriction on K is that the characteristic be zero; the above doesn't work in positive characteristic. For instance, suppose $\operatorname{char} K = p > 0$, and consider

$${v_1}^T = (\overbrace{1, \dots, 1}^{p}, \overbrace{0, \dots, 0}^{p}), {v_2}^T = (\overbrace{0, \dots, 0}^{p}, \overbrace{1, \dots, 1}^{p}).$$

Then $G(v_1, v_2) = \det \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \right) = 0$, but $\{v_1, v_2\}$ is clearly linearly independent. The problem is that nonzero vectors can have zero norms; worse, there are non-trivial subspaces (such as the span of $\{v_1, v_2\}$) in which the inner product of any two vectors is zero. This prevents us from naively following our geometric intuition from euclidean space, and is one of the reasons that the study of quadratic forms (generalized inner products) over positive characteristic is interesting.

Literature Cited

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[3] In particular, it is real; this eliminates the need to take the absolute value to obtain the volume. This is more significant in the complex case, where it shows that there is a well-defined notion of real-valued volume.