Let C be a subset of $\mathbb{R}^n, v \in C$. In class we gave three criteria for C to be a smooth curve at $\bar{v} = (v_1, \ldots, v_n)$.

First definition:[graph form] there exists $i, 1 \leq i \leq n$ an open subset U of $\mathbb{R}, v_i \in U$ and continuously differentiable functions $f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_n$ such that $C = \{f_1(u), \ldots, f_{i-1}(u), u, f_{i+1}(u), \ldots, f_n(u)\}, u \in U$ near $\bar{v}$.

Second definition:[level curve] there exists an open subset subset U of $\mathbb{R}^n, v \in U$ and continuously differentiable function $f \colon U \to \mathbb{R}^{n-1}$ such that $Df(\bar{v})$ is onto and $C = \{\bar{x} \in U | f(\bar{x}) = f(\bar{v})\}$ near $\bar{v}$.

Third definition:[parametrized curve] there exists a smooth map $r \colon I \to \mathbb{R}^n$ where I is an open subset of $\mathbb{R}, t_0 \in I$ such that

a)$r(t_0) = \bar{v}$,

b)$Dr(t_0) \neq 0$ and

c)$r \colon I \to \mathbb{R}^n$ is an embedding.

1)a)Prove that all three definitions are equivalent.

b)Give three definitions for the tangent line $T_C(\bar{v})$ to C at $\bar{v}$ [that is, give a definition in the case when C is given in the graph form, for the case when C is given as a level curve and for the case when C is given as a parametrized curve] and show the equivalence of these definitions [that is, show that the tangent line $T_C(\bar{v})$ does not depend on a way you describe the curve C].

c) Show that one can omit the condition that $r \colon I \to \mathbb{R}^n$ is an embedding in the third definition of smoothness.

2)Let $C\{(x,y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$.

a)Prove that for any $(x,y) \in C$, the curve C is smooth at (x,y) using the first definition of smoothness.

b)Find a parametrization $r = (r_1, r_2) \colon \mathbb{R} \to C - (-1, 0)$ such that $r_2(t) = t(r_1(t) + 1), t \in \mathbb{R}$

c)Find all pairs a, b of rational numbers such that $a^2 + b^2 = 1$.

d)Use the parametrization r of C to find $\int 1/\sqrt{1 - x^2} dx$

3)a)Let $f \colon \mathbb{R} \to \mathbb{R}$ be a twice differentiable function,such that $f(0) = 0$ and $f'(0) \neq 0$. As we know that exists an inverse function $g \colon I \to \mathbb{R}, g(0) = 0$ for a sufficiently small interval $I, 0 \in I$. Prove that g is twice differentiable at 0 and find the expression for $g''(0)$ in terms of $f'(0)$ and $f''(0)$.

b)Let $f \colon \mathbb{R} \to \mathbb{R}$ be a thrice differentiable function, such that $f(0) = 0$ and $f'(0) \neq 0$. As we know that exists an inverse function $g \colon I \to \mathbb{R}, g(0) = 0$ for a sufficiently small interval $I, 0 \in I$. Prove that g is thrice differentiable at 0 and find the expression for $g'''(0)$ in terms of $f'(0), f''(0)$ and $f'''(0)$.