Let $(V, \|v\|_V)$ be finite-dimensional normed vector space, $C \subset V$ a closed subset and $f : C \to \mathbb{R}$ a function. We define a function ω_f on the interval $(0, \infty)$ by $\omega_f(\delta) := sup|f(c) - f(c')|, c, c' \in C, \|c - c'\|_V < \delta$.

1. a)Prove that ω_f is a nondecreasing function on $(0, \infty)$.

b)Show that f is *uniformely continuous* iff [if and only if]

$$\lim_{\delta \to 0^+} \omega_f(\delta) = 0$$

c)Check whether the function $\sin(x^2)$ on $\mathbb{R}$ is uniformely continuous.

Let $f : C \to \mathbb{R}$ a continuous function. Fix $\epsilon > 0$. For any $c \in C$ we define $I_c \subset (0, 1)$ the set of all numbers x such that $\forall c' \in C$ such that $\|c - c'\|_V < x$ we have $|f(c) - f(c')| < \epsilon$. We define a function δ on C by $\delta(c) := lub(I_c)$.

2. Prove that the function δ on C is continuous and $\delta(c) > 0 \forall c \in C$.

Given a function f on $\mathbb{R}$ we say that f is *infinitely differentiable* if f is n-times differentiable for any $n \geq 0$. In this case for any $a \in \mathbb{R}$ we can define the *Taylor series*

$$P_f(a, x) := \sum_{n \geq 0} c_n(x - a)^n, c_n := f^{(n)}(a)/n!$$

. We define by $I_a \subset [0, \infty)$ as the set of numbers $r \geq 0$ such that $c_n r^n \to 0$ for $n \to \infty$ and write $r_f(a) := lub(I_a)$. [It could be that $r_f(a) = \infty$].

3. a)Show that the Taylor series $P_f(a, x)$ is convergent for any $x \in \mathbb{R}$ such that $|x - a| < r_f(a)$.

b)Find the Taylor series $P_f(a, x)$ for $f = 1/(x^2 + 1)$ and compute $r_f(a)$.

EXTRA CREDIT. Explain the result.

c)Let f be function f on $\mathbb{R}$ such that $f(x) = 0$ $x \leq 0$ and $f(x) = \exp(-1/x)$ for $x > 0$. Prove that f is infinitely differentiable and find $P_f(0, x)$.

d) Show that there exists an infinitely differentiable function f on $\mathbb{R}$ such that $0 \leq f(x) \leq 1, f(x) = 0$ for $|x| > 1$ and $f(x) = 1$ for $|x| < 1/2$

4. Let $f(x), g(x)$ be differentiable functions on $\mathbb{R}$ such that $g(x) \to +\infty$ for $x \to +\infty$ and such that there exists the limit $\lim_{x \to +\infty} f'(x)/g'(x)$. Show that in this case the limit $\lim_{x \to +\infty} f(x)/g(x)$ there exists and is equal to $\lim_{x \to +\infty} f'(x)/g'(x)$.

5. Prove that for any $x \geq 0$ there exists $\theta(x), 1/4 \leq \theta(x) \leq 1/2$ such that $\sqrt{x + 1} - \sqrt{x} = \frac{1}{2\sqrt{x + \theta(x)}}$.

6. Find coefficients a, b such that $\lim_{x \to 0}[x - (a + bcos(x))sin(x)]/x^4 = 0$

7. Let $U \subset \mathbb{R}$ be an open set such that $\mathbb{R} - U$ is also open. Prove that either $U = \mathbb{R}$ or U is empty..

8. a)Let C be a compact set $X_a \subset C, a \in A$ be a family of closed subsets of C such that for any finite subset $\{a_1, ..., a_N\} \subset A$ the intesection $\cap_{1 \leq i \leq N} X_{a_i}$ is not empty. Show that the intersection $\cap_{a \in A} X_a$ is also not empty.

b)Construct a family of closed subsets of $\mathbb{R}$ such that for any finite subset $\{a_1, ..., a_N\} \subset A$ the intesection $\cap_{1 \leq i \leq N} X_{a_i}$ is not empty but the intersection $\cap_{a \in A} X_a$ is empty.