HOMEWORK FOR DEC. 13-th

In this homework the letters V, W will denote finite dimensional vector spaces. For any vector space V we denote by $V^\vee$ the dual linear space. By the definition $V^\vee$, as a set, is the set of linear functionals on V. If $e_1, ..., e_n$ is a basis of V we denote by $v^i \in V^\vee, 1 \leq i \leq n$ the linear functionals on V such that $v^i(v_j) = \delta^i_j$ where $\delta^i_j = 1$ if $i = j$ and 0 otherwise.

A bilinear form f on $V \times W$ is a function $f : V \times W \to \mathbb{R}$ such that for any $v_0 \in V$ the function $w \to f(v_0, w)$ on W is linear and for any $w_0 \in W$ the function $v \to f(v, w_0)$ on V is linear. We denote the vector space of bilinear forms on $V \times W$ by $\mathcal{B}(V, W)$. Also for any pair V, W of vector spaces we denote, as before, the vector space of linear maps from V to W by $Hom(V, W)$.

1. Let $e_1, ..., e_n$ be is a basis of V and $f_1, ..., f_m$ be is a basis of W.

a)Show that for any pair $(i, j), 1 \leq i \leq n, 1 \leq j \leq m$ there exists unique bilinear form $b^{i,j}$ on $V \times W$ such that $b^{i,j}(e_{i'}, f_{j'}) = \delta^i_{i'}\delta^j_{j'}$.

b) Show that $b^{i,j}(v, w) = e^i(v)f^j(w)$.

c)Prove that $b^{i,j}, 1 \leq i \leq n, 1 \leq j \leq m$ is a basis of $\mathcal{B}(V, W)$.

2. Let P_n be the vector space of polynomials $p(t)$ in t of degree $\leq n$. For any $x \in \mathbb{R}$ we denote by $l_x \in P_n^\vee$ the linear functional on P_n such that $l_x(p) := p(x)$ for all $p = p(t) \in P_n$. Let $x_0, ..., x_n \in \mathbb{R}$ a set of distinct points.

a)Show that the set $\{l_{x_i}\} \in P_n^\vee, 0 \leq i \leq n$ of linear functionals is a basis of the space $P_n^\vee$.

b)Find a basis $e_0, ..., e_n$ in P_n such that $l_{x_i} = e^i, 0 \leq i \leq n$.

Let $f : V \times W \to \mathbb{R}$ be a bilinear form. By the definition for any $v \in V$ the function $f_v : W \to \mathbb{R}, f_v(w) := f(v, w)$ on W is linear. So we can consider f_v as an element of the dual space $W^\vee$. In other words the correspondence $v \to f_v$ defines a map $\alpha_f : V \to W^\vee$.

3 a)Show that the map $\alpha_f : V \to W^\vee$ is linear.

b) Show that the map $\alpha : \mathcal{B}(V, W) \to Hom(V, W^\vee)$ given by $\alpha(f) := \alpha_f, f \in \mathcal{B}(V, W)$ is linear.

c)Prove that α is an isomorphism of vector spaces.

Let $T : V \to V$ be a linear map. To any $l \in V^\vee$ we associate an function $T^\vee(l)$ on V defined by $T^\vee(l)(v) := l(T(v))$.

4. a) Show that the function $T^\vee(l)$ on V is linear.

b) By a) we can consider $T^\vee(l)$ as an element of $V^\vee$. Show that the map $T^\vee : V^\vee \to V^\vee, l \to T^\vee(l)$ is linear. We call it *the adjoint of T* c)Prove that for any linear maps $T, S : V \to V$ we have $(T \circ S)^{\vee} = S^{\vee} \circ T^{\vee}$.

d)Let $e_1, ..., e_n$ be is a basis of V and $A := A_T$ be the matrix of T in respect to this basis. Find the the matrix of $T^{\vee}$ in respect to the basis $e^1, ..., e^n$ of $V^{\vee}$.

e) Let $T : V \to W$ be a linear map. Define the the adjoint of T.