1. a)Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ Show that for any 3×3 matrix B such that $AB = BA$ we can find $c_0, c_1, c_2 \in \mathbb{R}$ such that

$$B = c_0 Id + c_1 A + c_2 A^2$$

Proof. If you write the matrix elements of matricies AB and BA you find that $AB = BA$ iff there exists a sequence $c_k, 0 \leq k \leq 2$ such that $b_{ij} = 0$ for $i > j$ and $b_{ij} = c_{j-i}$ for $i \leq j$. Then it is easy to see that

$$B = c_0 Id + c_1 A + c_2 A^2$$

b)Generalize the statement [and IF possible the proof] to the case of $n \times n$ matricies.

Let $A = (a_{ij})$ be an $n \times n$ matrix such that $a_{ij} = 1$ if $j - i = 1$ and $a_{ij} = 0$ otherwise. Let B be an $n \times n$ matrix such that $AB = BA$. Then there exists a sequence $c_k, 0 \leq k \leq n - 1$ such that

$$B = \sum_{0 \leq i \leq n-1} c_i A^i$$

Proof. If you write the matrix elements of matricies AB and BA you find that $AB = BA$ iff there exists a sequence $c_k, 0 \leq k \leq n - 1$ such that $b_{ij} = 0$ for $i > j$ and $b_{ij} = c_{j-i}$ for $i \leq j$. Then it is easy to see that

$$B = \sum_{0 \leq i \leq n-1} c_i A^i$$

2.Let V be the set of sequences $\bar{v} = (v_n), 1 \leq n < \infty$ such that $v_{n+2} = v_{n+1} + v_n, 1 \leq n < \infty$. We define the operations of addition and the scalar multiplication on V by

$\bar{v}' + \bar{v}'' := \bar{v}, \bar{v} := (v_n' + v_n''), 1 \leq n < \infty$ for $\bar{v}' = (v_n'), \bar{v}'' = (v_n'')$

$c\bar{v} = (cv_n), 1 \leq n < \infty$ for $\bar{v} = (v_n)$

a) Show that V is a finite-dimensional vector space and find it's dimension.

I'll leave for you to check the axioms.

Here are two ways to find the dimension of V.

Consider the map $R : V \to \mathbb{R}^2$ given by $R(\bar{v}) := (v_1, v_2)$. One can easily see that R is an isomorphism of vector spaces. Therefore $dim(V) = 2$.

Another proof. Let $\bar{v}' = (v_n') \in V$ the sequence such that $v_1' = 1, v_2' = 0, [v_3' = v_1' + v_2' = 1, ...]$ and $\bar{v}'' = (v_n'') \in V$ the sequence such that $v_1'' = 0, v_2'' = 1$. It is easy to see that $B := (\bar{v}', \bar{v}'')$ is a basis of V. So $dim(V) = 2$.

b) For any sequence $\bar{v} = (v_n) \in V$ we denote by $T\bar{v}$ the sequence $T\bar{v} := \bar{w}$ where $\bar{w} = (w_n), 1 \leq n < \infty, w_n := v_{n+1}$.

Show that for any $v \in V$ we have $T\bar{v} \in V$,

that the operator $T : V \to V$ is linear and

write a matrix A_T^B for T in a basis B of V [choose one].

I'll leave for you to check that T is linear. It is clear that $T\bar{v}' = \bar{v}'', T\bar{v}'' = \bar{v}' + \bar{v}''$. So the operator T in the basis B is given by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

3.Let V be a vector space of dimension $5, V', V'' \subset V$ subspaces such that $dimV' = dimV' = 3$. Show that $V' \cap V'' \neq \{\bar{0}\}$.

Proof. We assume that $V' \cap V'' = \{\bar{0}\}$ and show that this assumption leads to a contrudiction.

Let (e_1', e_2', e_3') be a basis of V' and (e_1'', e_2'', e_3'') be a basis of V''.

Claim. If $V' \cap V'' = \{\bar{0}\}$ then 6 vectors $(e_1', e_2', e_3', e_1'', e_2'', e_3'')$ are linearly independent.

Proof of the claim. Let $(c_1', c_2', c_3', c_1'', c_2'', c_3'') \in \mathbb{R}$ be numbers such that $c_1'e_1' + c_2'e_2' + c_3'e_3' + c_1''e_1'' + ce_2'' + ce_3'' = \bar{0}$. Then

$$c_1'e_1' + c_2'e_2' + c_3'e_3' = -[c_1''e_1'' + ce_2'' + c_3e_3'']$$

. But the right side belongs to V' and the left side side belongs to V''. Since $V' \cap V'' = \{\bar{0}\}$ we see that $c_1'e_1' + c_2'e_2' + c_3'e_3' = -[c_1''e_1'' + ce_2''e_2'' + ce_3''e_3''] = \bar{0}$. Since (e_1', e_2', e_3') is a basis of V' and (e_1'', e_2'', e_3'') is a basis of V'' we see that $(c_1', c_2', c_3', c_1'', c_2'', c_3'') = (0, 0, 0, 0, 0, 0)$. We see that $(e_1', e_2', e_3', e_1'', e_2'', e_3'')$ are linearly independent.

Since $dimV = 5$ we know that one can not find 6 linearly independent vectors in V. The contrudiction. So our assumption that $V' \cap V'' = \{\bar{0}\}$ was wrong and $V' \cap V'' \neq \{\bar{0}\}$.

4. Let $S^1 := \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$.

a) Show that the set $S^1 \subset \mathbb{R}^2$ is compact. Proof one easily checks that S^1 is bounded and that the complement $\mathbb{R}^2 - S^1$ is open. So S^1 is bounded and closed and therefore compact.

b) Show that S^1 is connected [that is show that for any closed subset X of S^1 such that the set $S^1 - X$ is closed we either have $X = \emptyset$ or $X = S^1$.

Consider the map $f : \mathbb{R} \to S^1$ given by

$$\theta \to (\cos(\theta), \sin(\theta)), \theta \in \mathbb{R}$$

It is easy to see that f is continuous. Therefore for any closed subset $Y \subset S^1$ the preimage $f^{-1}(Y) \subset \mathbb{R}$ is also closed.

Let X be a closed subset X of S^1 such that the set $S^1 - X$ is also closed, $Z := f^{-1}(X) \subset \mathbb{R}$. Then $\mathbb{R} - Z = f^{-1}(S^1 - X)$. Since X and $S^1 - X$ are closed we see that Z and $\mathbb{R} - Z$ are also closed. But we know [a mid-term exam] that this is possible only if either $Z = \emptyset$ or $Z = \mathbb{R}$. But in this case either have $X = \emptyset$ or $X = S^1$.

5. Let f be a function on $\mathbb{R}$, $f(x) := \frac{sin(x^2)}{1+x^2}$.

Find whether f is a uniformely continuous function on $\mathbb{R}$ [and give a proof to justify the conclusion].

Here are two ways to prove that f is a uniformely continuous.

The first proof. You check that the derivative $f'(x)$ is bounded. That is there exists $C > 0$ such that $|f'(z)| < C$ for all $z \in \mathbb{R}$.

Now we prove that f is a uniformely continuous. For this we have to show that for any $\epsilon >$ there esists $\delta > 0$ such that for all $x, y \in \mathbb{R}, |x - y| < \delta$ we have $|f(x) - f(y)| \leq \epsilon$.

Take $\delta = \epsilon/C$. By the MVT for any $x, y \in \mathbb{R}$ there exists $z \in \mathbb{R}$ such that $f(x) - f(y) = f'(z)(x - y)$. Since $|f'(z)| < C$ for all $z \in \mathbb{R}$ we have $|f(x) - f(y)| \leq C|x - y|$. So if $|x - y| < \delta$ we have $|f(x) - f(y)| \leq C\delta = \epsilon$.

The second proof. We check that $f(z) \to 0$ for $|z| \to \infty$.

Now we prove that f is a uniformely continuous. For this we have to show that for any $\epsilon >$ there esists $\delta > 0$ such that for all $x, y \in \mathbb{R}, |x - y| < \delta$ we have $|f(x) - f(y)| \leq \epsilon$.

Since $f(z) \to 0$ for $|z| \to \infty$ we can find $N > 1$ such that $|f(z)| < \epsilon/2$ for $|z| > N - 1$. Now the interval $[-N, N]$ is compact. Since f is continuous it's restriction to the interval $[-N, N]$ is uniformely continuous. Therefore we can find $\delta > 0$ such that for all $x, y \in [-N, N], |x - y| < \delta$ we have $|f(x) - f(y)| \leq \epsilon$. We can assume that $\delta < 1$.

Now I claim that for all $x, y \in \mathbb{R}, |x - y| < \delta$ we have $|f(x) - f(y)| \leq \epsilon$. Really, since $|x - y| < \delta < 1$ either $x, y \in [-N, N]$ or $|x| > N - 1$ and $|y| > N - 1$.

In the first case [by the definition of δ] we have $|f(x) - f(y)| \leq \epsilon$.

In the second case we have $|f(x)| < \epsilon/2$ and $|f(y)| < \epsilon/2$ and therefore $|f(x) - f(y)| \leq \epsilon$.

6. Let V_n be the space of polynomials $p(x)$ such that degree $p \leq n$. Consider a function $F : V_{10} \to V_{20}$ given by $F(p) := p^2$.

a) Show that the function F is differentiable at all $p \in V$ where $V := V_{10}$.

and

b) Find $D_F(p) : V_{10} \to V_{20}$ for $p(x) \equiv 1$.

Proof. Let $W := V_{20}$. For any $p \in V_n$ we have $F(p + h) - F(p) = 2hp + h^2$. Define $L_P : V \to W$ by $L_P(h) := 2ph$. It is clear that L_P is a linear map and $|F(p + h) - F(p) - L_P(h)|/|h| = |h|^2/|h| \to 0$ for

$|h| \to 0$. So the function F is differentiable at all $p \in V$ and the linear map $D_F(p) : V_{10} \to V_{20}$ is given by $D_F(p)(h) = 2ph$.

c)Show that all partial derivatives of F of order 3 are equal to 0.

If you write F in coordinates $F = (F_i, 0 \leq i \leq 20), F_i = F_i(x_j), 0 \leq j \leq 10$ you will see that all the functions $F - i$ are quandratic polynomials. Therefore all partial derivatives of F of order 3 are equal to 0.

7. Let f be a function on $\mathbb{R} - \{-1\}$ given by $f(x) := \ln(1 + x)$ and let $p_f^n(1)(x) := \sum_{0 \leq i \leq n} \frac{f^{(i)}(1)}{i!}(x-1)^i$ be the Taylor polynomial for f at the point 1.

Show that $|p_f^n(1)(0)| < 1/2^n$.

Proof. We first show that the Taylor series $p_f^n(1)(x) := \sum_{0 \leq i < \infty} \frac{f^{(i)}(1)}{i!}(x-1)^i$ is convergent for $x = 0$. For this we have to show that $R_n(0) \to 0$ for $n \to \infty$ where $R_n(x)$ is the remainder. By the formula for the remainder [see p.182] we have $R_n(0) = \frac{f^{(n+1)}(c)}{(n+1)!}(c-1)^{(n+1)}$ for some $c, 0 \leq c \leq 1$. By induction you can show that

$$\frac{f^(n+1)(c)}{(n+1)!} = (-1)^n (1+c)^{-(n+1)/(n+1)}, n \geq 0$$

. So $R_n(0) \to 0$ for $n \to \infty$.

We have $f(0) = \ln(1) = 0$ and therefore

$$R_n(0) = f(0) - p_f^n(1)(x) = \sum_{n+1 \leq i < \infty} \frac{f^{(i)}(1)}{i!}(x-1)^i = \sum_{n \leq i < \infty} (-1)^n 2^{-(n+1)}/(n+1)$$

Now it is easy to show that $|R_n(0)| < 1/2^n$

8. Let $u(x, y)$ be a differentiable function on $\mathbb{R}^2$ such that $u(x, x^2) = 1$ for all $x \in \mathbb{R}$. Suppose that $\partial u/\partial x(2, 4) = 1$. Find $\partial u/\partial y(2, 4)$.

Let $f(x) := u(x, x^2)$. Compute $\partial f/\partial x$ in two ways. First of all $f(x) \equiv 1$. So $\partial f/\partial x \equiv 0$. On the other hand we can apply the chain rule. Then we obtain

$$\partial f/\partial x(a) = \partial u/\partial x(a, a^2) + 2a\partial u/\partial y(a, a^2)$$

In particular $\partial f/\partial x(2) = \partial u/\partial x(2, 4) + 4\partial u/\partial y(2, 4)$. So $0 = 1 + 4\partial u/\partial y(2, 4)$. Or $\partial u/\partial y(2, 4) = -1/4$

9. a)Let $B : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be an antisymmetric bilinear form B such that B is not identically zero. [That is there exists a pair $(\bar{v}, \bar{w})$ of vectors in $\mathbb{R}^2$ such that $B(\bar{v}, \bar{w}) \neq 0$]. Show that there exists a basis f_1, f_2 such that $B(f_1, f_2) = 1$.

Let $(\bar{v}, \bar{w})$ be vectors in $\mathbb{R}^2$ such that $B(\bar{v}, \bar{w}) = r \neq 0$. Take $f_1 := \bar{v}, f_2 := \bar{w}/r$. Then $B(f_1, f_2) = 1$.

Claim. (f_1, f_2) are linearly independent.

Proof. Suppose that there exist $a_1, a_2 \in \mathbb{R}$ such that $a_1 f_1 + a_2 f_2 = \bar{0}$. Then we have $0 = B(\bar{0}, f_1) = B(a_1 f_1 + a_2 f_2, f_1) = -a_2$. So $a_2 = 0$. Analogously one shows that $a_1 = 0$. Claim is proved.

remark. Since $dim\mathbb{R}^2 = 2$ the linear independence of (f_1, f_2) implies that (f_1, f_2) is a basis of $\mathbb{R}^2$.

b)Let $B : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be an antisymmetric bilinear form B such that B is not identically zero. Show that there exists a basis f_1, f_2, f_3 in $\mathbb{R}^3$ such that $B(f_1, f_2) = 1, B(f, f_3) = 0$ for all $f \in \mathbb{R}^3$

As before we can find $f_1, f_2 \in \mathbb{R}^3$ such that $B(f_1, f_2) = 1$. Consider now a pair of linear equations on $v \in \mathbb{R}^3$ given by $B(f_1, v) = 0, B(f_2, v) = 0$. Since $dim\mathbb{R}^3 = 3 > 2$ this system of equations have a nonzero solution $v \neq \bar{0}$.

Claim. v is not a linear combination of f_1, f_2.

Proof of Claim. Assume that $v = a_1 f_1 + a_2 f_2$. Then $B((f_1, v) = a_1$ and $B((f_2, v) = a_2$ So $a_1 = a_2 = 0$. But by the construction $v \neq \bar{0}$.

Since v is not a linear combination of f_1, f_2 and [as we have seen before] f_1, f_2 are linearly independent three vectors $f_1, f_2, v \in \mathbb{R}^3$ are linearly independent (?). Since $dim(\mathbb{R}^3) = 3$ we see that (f_1, f_2, v) is a basis of $\mathbb{R}^3$.

By the construction $B(f_1, v) = 0, B(f_2, v) = 0$. since B is antilinear we have $B(v, v) = 0$. Therefore (?) $B(f, v) = 0$ for all $f \in \mathbb{R}^3$ and we can take $f_3 = v$.