1. a)Prove that ω_f is a nondecreasing function on $(0, \infty)$.

If $\delta' < \delta$ then for any pair c, c' such that $\|c - c'\|_V < \delta'$ we have $\|c - c'\|_V < \delta$. Therefore $sup|f(c) - f(c')|, c, c' \in C, \|c - c'\|_V < \delta' \leq sup|f(c) - f(c')|, c, c' \in C, \|c - c'\|_V < \delta$ So $\omega_f(\delta') \leq \omega_f(\delta)$.

b)Show that f is uniformely continuous iff [if and only if]

$$\lim_{\delta \to 0^+} \omega_f(\delta) = 0$$

By the definition, a function f is uniformely continuous iff for any $\epsilon > 0$ there exists $\delta > 0$ such that for any $c, c' \in C, \|c - c'\|_V < \delta$ we have $|f(c) - f(c')| < \epsilon$.

Assume that $\lim_{\delta \to 0^+} \omega_f(\delta) = 0$ and show that f is uniformely continuous . Choose any $\epsilon > 0$. Since $\lim_{\delta \to 0^+} \omega_f(\delta) = 0$ we can find $\delta > 0$ such that $\omega_f(\delta) < \epsilon$. Therefore for any $c, c' \in C, \|c - c'\|_V < \delta$ we have $|f(c) - f(c')| < \epsilon$ and we see that f is uniformely continuous.

Conversely, assume that f is uniformely continuous. To show that $\lim_{\delta \to 0^+} \omega_f(\delta) = 0$ it is suffiicient to show that for any $\epsilon > 0$ there exists $\delta > 0$ such that such $\omega_f(\delta') < \epsilon$ for $\delta' < \delta$. By the definition of the uniformely continuity there exists $\delta > 0$ such that such for any $c, c' \in C, \|c - c'\|_V < \delta$ we have $|f(c) - f(c')| < \epsilon$. But this implies that $\omega_f(\delta) < \epsilon$. Since , by a), the function $\omega_f(\delta)$ is nondecreasing we see that $\omega_f(\delta') < \epsilon$ for $\delta' < \delta$.

c)Check whether the function $\sin(x^2)$ on $\mathbb{R}$ is uniformely continuous.

Choose $\epsilon = 1$. It is easy to see that for any $\delta > 0$ there exists a positive integer n such that $x_{2n+1} - x_{2n-1} < \delta$ where $x_n := \sqrt{n\pi}$. On the other hand $\sin(x_{2n+1}^2) - \sin(x_{2n-1}^2) > 1$. Therefore the function $\sin(x^2)$ on $\mathbb{R}$ is not uniformely continuous.

For the simplicity I'll restate the problem 2 in a slightely different form. Let $f : C \to \mathbb{R}$ a continuous function. Fix $\epsilon > 0$. For any $c \in C$ we define $I_c \subset (0, 1)$ the set of all numbers x such that $\forall c', c'' \in C$ such that $\|c - c'\|_V, \|c - c''\|_V < x$ we have $|f(c'') - f(c')| < \epsilon$. We define a function δ on C by $\delta(c) := lub(I_c)$.

2. Prove that the function δ on C is continuous and $\delta(c) > 0 \forall c \in C$.

Proof. The positivity of the function δ follows immediately from the contnuity of the function f.

We will actually prove that the function δ on C is uniformely continuous. More precisely we show that for any $\nu > 0$ we have $|\delta(c) - \delta(d)| < \nu$ for any $c, d \in C, \|c - d\|_V < \nu/2$. In other words we have show that $\delta(c) < \delta(d) + \nu$ and that $\delta(d) < \delta(c) + \nu$ for any $c, d \in C, \|c - d\|_V < \nu/2$. Of course, it is sufficient to prove the first inequality $\delta(d) < \delta(c) + \nu$. It is clear that $B_d(\delta(c) - \nu) \subset B_c(\delta(c))$. Since, by the definition of $\delta(c)$, we have $|f(c'') - f(c')| < \epsilon \forall c', c'' \in B_c(\delta(c))$ we see that

$|f(c'') - f(c')| < \epsilon \forall c', c'' \in B_d(\delta(c) - \nu)$. Therefore $\delta(d) > \delta(c) - \nu$. So $\delta(c) < \delta(d) + \nu$.

Given a function f on $\mathbb{R}$ we say that f is *infinitely differentiable* if f is n-times differentiable for any $n \geq 0$. In this case for any $a \in \mathbb{R}$ we can define the *Taylor series*

$$P_f(a,x) := \sum_{n \geq 0} c_n (x-a)^n, c_n := f^{(n)}(a)/n!$$

. We define by $I_a \subset [0, \infty)$ as the set of numbers $r \geq 0$ such that $c_n r^n \to 0$ for $n \to \infty$ and write $r_f(a) := lub(I_a)$. [It could be that $r_f(a) = \infty$].

a)Show that the Taylor series $P_f(a,x)$ is convergent for any $x \in \mathbb{R}$ such that $|x-a| < r_f(a)$.

We have to show that for any $x \in \mathbb{R}$ such that $|x-a| < r_f(a)$ the series $\sum_{n \geq 0} c_n (x-a)^n$ is convergent. For simplicity of writing we assume that $x > a$. Choose any $y \mathbb{R}$ such that $x < y < a + r_f(a)$. Since $y < a + r_f(a)$ we see from the definition of $r_f(a)$ that $\lim_{n \to \infty} c_n (y-a)^n = 0$. In particular there exists a number $C > 0$ such that $|c_n (y-a)^n| < C$ forall $n > 0$. Therefore $|c_n (x-a)^n| < C(x-a)^n/(y-a)^n$ and the Taylor series $P_f(a,x)$ is majorated by a convergent geometric series. Therefore the Taylor series $P_f(a,x)$ is convergent.

b)Find the Taylor series $P_f(a,x)$ for $f = 1/(x^2+1)$ and compute $r_f(a)$.

By the definition we have to find coefficients $c_n := f^{(n)}(a)/n!$. The easiest way to find the derivatives $f^{(n)}(a)$ is to use the decomposition

$f(x) = i/2[1/(x+i) - 1/(x-i)]$ where $i^2 = 1$. Then one finds that $f^{(n)}(x) = (-1)^n n! i/2[1/(x+i)^n - 1/(x-i)^n]$ and

$c_n = (-1)^n i/2[1/(a+i)^n - 1/(a-i)^n] = (-1)^n \sum_k (-1)^k \binom{n}{2k}$.

Another way to get a formula for $P_f(a,x)$ is to compute $f^{(n)}(a)/n!$ for small n by direct computations, to make a guess for $f^{(n)}(a)/n!$ for arbitrary n and then to prove the guess by induction.

EXTRA CREDIT. Explain the result.

Since the function $1/(x^2+1)$ is not defined at $x = \pm i$ we have $< r_f(a) = \sqrt{a^2+1} =$ the distance from a to i.

c)Let f be function f on $\mathbb{R}$ such that $f(x) = 0$ $x \leq 0$ and $f(x) = \exp(-1/x)$ for $x > 0$. Prove that f is infinitely differentiable and find $P_f(0,x)$.

Proof. We start with the following result.

Lemma'. For any $n \leq 0$ we have $\lim_{x \to 0^+} x^{-n} \exp(-1/x) = 0$.

Proof of the Lemma. It is clear that

$$\lim_{x\to 0^+} x^{-n}\exp(-1/x) = \lim_{y\to\infty} y^n \exp(-y)$$

and one can find this limit using the L'Hopital's rule.

Next one proves the following result.

Lemma". For any $n \leq 0$ there exists a polynomial $P_n(t)$ of degree n such that $\exp(-1/x)^{(n)} = P_n(1/x)\exp(-1/x)$.

Proof of Lemma. Induction in n.

Now the part c) follows immediately from these Lemmas. In this case we have $P_f(0,x) = 0$.

d) Show that there exists an infinitely differentiable function F on $\mathbb{R}$ such that $0 \leq F(x) \leq 1, F(x) = 0$ for $|x| > 1$ and $F(x) = 1$ for $|x| < 1/2$. I think that this problem was too difficult. I should ask you first to construct an infinitely differentiable function g on $\mathbb{R}$ such that $0 \leq g(x) \leq 1, g(x) = 0$ for $|x| \geq 1$ and $g(x) > 0$ for $|x| < 1$. In this case we can simply take $g(x) := f(x+1)f(1-x)$ where f is the fuction from the part c).

To construct a function $F(x)$ satisfies the conditions of d) we define $a := min_{|x|\leq 1/2} g(x)$. Then $a > 0$ and therefore $f(a) > 0$. Consider now the function $F(x) := 1 - f(a - g(x))/f(a)$. It is easy to check that $F(x)$ satisfies the conditions of d)

4. Let $f(x), g(x)$ be differentiable functions on $\mathbb{R}$ such that $g(x) \to +\infty$ for $x \to +\infty$ and such that there exists the limit $\lim_{x\to+\infty} f'(x)/g'(x)$. Show that in this case the limit $\lim_{x\to+\infty} f(x)/g(x)$ exists and is equal to $\lim_{x\to+\infty} f'(x)/g'(x)$.

Since the $\lim_{x\to+\infty} f'(x)/g'(x)$ exists we know that there exists $a \in \mathbb{R}$ such that $g'(x) \neq 0$ for $x > a$.

Let $A = \lim_{x\to+\infty} f'(x)/g'(x)$. We want to show that $\lim_{x\to+\infty} f(x)/g(x) = A$. In other words we have to show that for any $\epsilon > 0$ there exist $b \in \mathbb{R}$ such that $|f(x)/g(x) - A| \leq \epsilon$ for $x > b$. Since $\lim_{x\to+\infty} f'(x)/g'(x) = A$ we can find $b \in \mathbb{R}$ such that $|f'(x)/g'(x) - A| < \epsilon$ for $x > b$. Choose now any $x > b$ and apply the Cauchy theorem to functions $f(x), g(x)$ in the interval $[b, C]$ where $C > b$. We see that there exists $y_C \in [b, C]$ such that $\frac{f(C)-f(b)}{g(C)-g(b)} = \frac{f'(y_C)}{g'(y_C)}$. Since $y_C \geq b$ we see that $|\frac{f'(y_C)}{g'(y_C)} - A| < \epsilon$. So

$$|\frac{f(C)-f(b)}{g(C)-g(b)} - A| < \epsilon$$

for any $C > b$. Therefore $|\frac{f(C)/g(C)-f(b)/g(C)}{1-g(b)/g(C)} - A| < \epsilon$ for any any $C > b$. In other words we see that the limit $\lim_{C\to\infty} \frac{f(C)/g(C)-f(b)/g(C)}{1-g(b)/g(C)}$ exists and is equal to A. Since, by the assumptions, $g(x) \to +\infty$ for

$x \to +\infty$ we see that the limit $\lim_{x\to+\infty} f(x)/g(x)$ exists and is equal to $\lim_{x\to+\infty} f'(x)/g'(x)$.

5. Prove that for any $x \geq 0$ there exists $\theta(x), 1/4 \leq \theta(x) \leq 1/2$ such that $\sqrt{x+1} - \sqrt{x} = \frac{1}{2\sqrt{x+\theta(x)}}$.

Proof. We have $\sqrt{x+1} - \sqrt{x} = \frac{1}{\sqrt{x+1}+\sqrt{x}}$. Therefore $\theta(x)$ sutisfies the equation

$$\sqrt{x+1} + \sqrt{x} = 2\sqrt{x+\theta(x)}$$

or

$$2x + 1 + 2\sqrt{x(x+1)} = 4(x+\theta)$$

. Since the right side is a monotonely increasing function of θ to show that $1/4 \leq \theta(x) \leq 1/2$ it is sufficient to check that

a)$2x + 1 + 2\sqrt{x(x+1)} > 4(x+1/4)$ and

b)$2x + 1 + 2\sqrt{x(x+1)} > 4(x+1/2)$.

But a) is equivalent to an inequality $x < \sqrt{x(x+1)}$ and b) is equivalent to an inequality $x+1 > \sqrt{x(x+1)}$. But both inequalities follow from the obvious inequality $x < x+1$.

6. Find coefficients a, b such that $\lim_{x\to 0}[x-(a+bcos(x))sin(x)]/x^4 = 0$.

Let $f(x) := x - (a + bcos(x))sin(x)$. If $\lim_{x\to 0} f(x)/x^4 = 0$ then it follows from the L'Hopital's rule that that a, b should be such that $\lim_{x\to 0} f'(x)/x^3 = 0, \lim_{x\to 0} f''(x)/x^2 = 0, \lim_{x\to 0} f'''(x)/x = 0$ and $\lim_{x\to 0} f^{(4)}(x) = 0$. In particular a, b should be such $f'(0) = f''(0) = f'''(0) = f^{(4)}(0) = 0$. This gives a system of linear equations on (a, b) which implies that $a = 4/3, b = -1/3$.

There is a faster way to find (a, b). One can use the Taylor polynomials $P^3(0, x)$ for $sin(x)$ and $cos(x), P^3_{sin}(0, x) = x - x^3/6, P^3_{cos}(0, x) = 1 - x^2/2$. This shows that $f(x) = x(1 - x^2/6)(a + b - bx^2/2) + R(x)$ where $\lim_{x\to 0} R(x)/x^4 = 0$. Therefore $a + b = 1, -b/2 = 1/6$. This implies $a = 4/3, b = -1/3$.

7. Let $U \subset \mathbb{R}$ be an open set such that $\mathbb{R} - U$ is also open. Prove that either $U = \mathbb{R}$ or U is empty.

Assume that both U and $\mathbb{R} - U$ are not empty and show that this assumption leads to a contrudiction.

Let $a \in U, b \in \mathbb{R} - U$. Then either $a < b$ or $b < a$. Assume that $a < b$. Let $X := [a, b] \cap U$. Then X is a nonempty bounded subset of $\mathbb{R}$ and we can define $x := lub(X)$. Let Since $\mathbb{R} - U$ is open U is closed. Therefore X is closed and $x \in$. Since U is open and $x \in U$ there exists $r > 0$ such that $(x - r, x + r) \subset U$. Therefore $x + r/2 \in U$. But this contrudict the definition $x := lub(X)$.

8. a)Let C be a compact set of normed vector space $V, X_a \subset C, a \in A$ be a family of closed subsets of C such that for any finite subset $\{a_1, ..., a_N\} \subset A$ the intesection $\cap_{1 \leq i \leq N} X_{a_i}$ is not empty. Show that the intersection $\cap_{a \in A} X_a$ is also not empty.

Proof. We assume that the intersection $\cap_{a \in A} X_a$ is empty and show that this assumption leads to a contrudiction.

Let $Y_a := V - X_a$. Then Y_a is an open cover of C. Since, by the assumption, C is compact we can find a finite set $\{a_1, ..., a_N\} \subset A$ such that $C \subset \bigcup_{1 \leq i \leq N} Y_{a_i}$. But this implies that the intesection $\cap_{1 \leq i \leq N} X_{a_i}$ is empty. A contrudiction.

b)Construct a family of closed subsets of $\mathbb{R}$ such that for any finite subset $\{a_1, ..., a_N\} \subset A$ the intesection $\cap_{1 \leq i \leq N} X_{a_i}$ is not empty but the intersection $\cap_{a \in A} X_a$ is empty.

Answer. Take $A = \mathbb{Z}, X_a := [a, +\infty]$