

If V is a vector space $B = \{e_1, \dots, e_d\}$ be a basis of V and $T : V \rightarrow V$ is a linear transformation we denote by $A_B(T)$ the matrix of T in the basis B . That is $A_B(T) = (a_j^i), 1 \leq i, j \leq d$ is a matrix such that $T(e_j) = \sum_{i=1}^d a_j^i e_i$.

1. a) Let $B_0 =: \{e_1, e_2\}$ be the standard basis of $\mathbb{R}^2, T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation such that

$$A_{B_0}(T) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

[that is

$$T(e_1) = 2e_1 + e_2, T(e_2) = e_1 + e_2]$$

Find a basis $B =: \{f_1, f_2\}$ of \mathbb{R}^2 such that the matrix $A_B(T)$ is diagonal. That is $A_B(T) = \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

A SOLUTION.

If $\{f_1, f_2\}$ is a basis of \mathbb{R}^2 such that the matrix $A_B(T)$ is diagonal then $Tf_1 = \lambda_1 f_1, Tf_2 = \lambda_2 f_2$. So to find such a basis $\{f_1, f_2\}$ we have to construct vectors $f \in \mathbb{R}^2$ such that $Tf = \lambda f, \lambda \in \mathbb{R}$.

Let $f = ae_1 + be_2$. Then $Tf = (2a+b)e_1 + (a+b)e_2$. Therefore $Tf = \lambda f$ iff $2a+b = \lambda a, a+b = \lambda b$. In this case we have $(2a+b)b = (a+b)a$ and therefore $x^2 + x - 1 = 0$ where $x := b/a$. So $x = 1/2[-1 \pm \sqrt{5}]$. So we can take $f_1 = e_1 + [-1/2 + \sqrt{5}/2]e_2, f_2 = e_1 + [-1/2 - \sqrt{5}/2]e_2$.

b) find the numbers λ_1 and λ_2 .

SOLUTIONS.

A) Using a) you see that $\lambda_1 = 2 + [-1/2 + \sqrt{5}/2] = 3/2 + \sqrt{5}/2, \lambda_2 = 2 + [-1/2 - \sqrt{5}/2] = 3/2 - \sqrt{5}/2$.

B) The second solution. You can find λ_1 and λ_2 directly. Assume that $\lambda \in \mathbb{R}$ is such that there exists $f \in \mathbb{R}^2 - \bar{0}$ such that $Tf = \lambda f$. Then $(T - \lambda Id)f = \bar{0}$. In other words $Ker(T - \lambda Id) \neq \bar{0}$. Therefore (?) $det(T - \lambda Id) = 0$. But

$$A_{B_0}(T - \lambda Id) = \begin{pmatrix} 2 - \lambda & 1 \\ 1 & 1 - \lambda \end{pmatrix}$$

and therefore $det(T - \lambda Id) = (2 - \lambda)(1 - \lambda) - 1 = \lambda^2 - 3\lambda + 1$ and $\lambda = 3/2 \pm \sqrt{5}/2$.

c) Let $f_1 = c_1^1 e_1 + c_1^2 e_2, f_2 = c_2^1 e_1 + c_2^2 e_2, c_j^i \in \mathbb{R}$.

Show that $A = C\Lambda C^{-1}$ where $A := A_{B_0}(T)$.

$$C = \begin{pmatrix} c_1^1 & c_1^2 \\ c_2^1 & c_2^2 \end{pmatrix}$$

A SOLUTION.

Let $S, \Lambda : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear maps such that $Se_1 = f_1, Se_2 = f_2$ and $\Lambda e_1 = \lambda_1 e_1, \Lambda e_2 = \lambda_2 e_2$. We have to show that

$$(*) \quad T = S\Lambda S^{-1}$$

Since both sides of $(*)$ are linear maps and $\{f_1, f_2\}$ is a basis of \mathbb{R}^2 it is sufficient to check that $Tf_1 = S\Lambda S^{-1}f_1$ and $Tf_2 = S\Lambda S^{-1}f_2$. We will check the first equality.

By the construction $Tf_1 = \lambda_1 f_1$. On the other hand since $Se_1 = f_1$ we have

$$S\Lambda S^{-1}f_1 = S\Lambda e_1 = S\lambda_1 e_1 = \lambda_1 Se_1 = \lambda_1 f_1$$

d) Let $a = (3 - \sqrt{5})/2$. Find the elements of the matrix $a^{1000} A^{1000}$ with the precision of 10 per cent.

A SOLUTION.

We have $A = C\Lambda C^{-1}$. Therefore $A^{1000} = C\Lambda^{1000}C^{-1}$ (?). So $a^{1000}A^{1000} = C(a\Lambda)^{1000}C^{-1}$. But

$$a\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & a^2 \end{pmatrix}$$

and therefore

$$a^{1000}A^{1000} = C \begin{pmatrix} 1 & 0 \\ 0 & a^{2000} \end{pmatrix} C^{-1}$$

. Since $|a| < 1/2$ the number a^{2000} is very small. Therefore with a very good approximation we have

$$a^{1000}A^{1000} \sim C \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} C^{-1} = \frac{1}{\sqrt{5}} \begin{pmatrix} (\sqrt{5} + 1)/2 & 1 \\ 1 & (\sqrt{5} - 1)/2 \end{pmatrix}$$

2.a) Let f be a differentiable function on \mathbb{R} such that $f'(x) \equiv 1$.

Prove that there exists $c \in \mathbb{R}$ such that $f(x) = x + c$ for all $x \in \mathbb{R}$.

SOLUTIONS. A) By the mean-value theorem we have $f(x) - f(0) = (x - 0)f'(y)$ for $0 \leq y \leq x$ [I assume that $x > 0$]. So $f(x) - f(0) = x$ for all $x \in \mathbb{R}$. Or $f(x) = c + x$ for $c = f(0)$.

B) The second solution.

It is clear that f is a twice differentiable function on \mathbb{R} . Therefore we can apply the Taylor formula

$f(x) = f(0) + f'(0)x + r_2(x)$ where $r_2(x) = 1/2 f''(y)$ for $0 \leq y \leq x$ [I assume that $x > 0$]. But $f'(y) = 1, f''(y) = 0$ for all $y \in \mathbb{R}$. So $f(x) = c + x$ for $c = f(0)$.

The second solution works in the case b)

b) Find all twice differentiable functions F on \mathbb{R} such that $F''(x) \equiv 1$.

Let $f : Mat(n, n) \rightarrow Mat(n, n)$ be a function such that $f(A) = A^2$.
 3. a) Show that the function f is differentiable and for any $n \times n$ matrix H find the differential $df(H) : Mat(n, n) \rightarrow Mat(n, n)$.

A SOLUTION.

Fix $A \in Mat(n, n)$ and consider $f(A + h)$, $h \in Mat(n, n)$. We have $f(A + h) = A^2 + D_A(h) + h^2$ where $D_A(h) = Ah + hA$. It is clear that $h \rightarrow D_A(h)$ is a linear map from $Mat(n, n)$ to $Mat(n, n)$ and $|h^2|/|h| \rightarrow 0$ for $h \rightarrow 0$. Therefore f is differentiable for all $A \in Mat(n, n)$ and $D_f(A)(h) = Ah + hA$.

We can identify the vector space $Mat(n, n)$ with \mathbb{R}^{n^2} , $A \rightarrow \{a_{i,j}\}$. Therefore we can define the partial derivatives $\partial f / \partial a_{i,j}$.

b) Find the partial derivatives $\partial f / \partial a_{i,j}$.

A SOLUTION. Let $E_{i,j} \in Mat(n, n)$ be the basis of matrices such the i, j -th entry of $E_{i,j}$ is equal to 1 and the all other entries are equal to zero. This basis defines the identification of the vector space $Mat(n, n)$ with \mathbb{R}^{n^2} , $A \rightarrow \{a_{i,j}\}$. Therefore $\partial f / \partial a_{i,j} = D_f(A)(E_{i,j}) = \sum_{1 \leq k \leq n} a_{i,k} E_{k,j} + \sum_{1 \leq k \leq n} a_{k,j} E_{i,k}$

c) Find the differential $dF(H) : Mat(n, n) \rightarrow Mat(n, n)$ where $F(A) := A^3$.

A SOLUTION. As in a) one can show that $D_f(A)(h) = A^2h + AhA + A^2h$.

4. a) Let f be a differentiable function on \mathbb{R} and F be and function on \mathbb{R}^2 defined by $F(x, y) := x^7 f(y/x^2)$.

Show that $x \partial F / \partial x + 2y \partial F / \partial y = 7F$

b) Find a generalization of this result.

If f is an infinitely differentiable function on \mathbb{R} at $a \in \mathbb{R}$ we can define the Taylor series

$$P_f(a, x) := \sum_{n \geq 0} c_n (x - a)^n, c_n := f^{(n)}(a) / n!$$

. We define by $I_a \subset [0, \infty)$ as the set of numbers $r \geq 0$ such that $c_n r^n \rightarrow 0$ for $n \rightarrow \infty$ and write $r_f(a) := \text{lub}(I_a)$. [It could be that $r_f(a) = \infty$].

Let f be a function on $\mathbb{R} - \{0, 1, 2\}$ given by $f(x) := \frac{1}{x(x-1)(x-2)}$.

5. a) Show that f is an infinitely differentiable function on $\mathbb{R} - \{0, 1, 2\}$

A SOLUTION. One can prove the following general result. Let F be an infinitely differentiable function on \mathbb{R} $Z \subset \mathbb{R}$ the set of zeros of F . Then the function $1/F$ on $\mathbb{R} - Z$ is infinitely differentiable. Then one applies this result to the case when $F = x(x - 1)(x - 2)$.

b) Find $r_f(a)$ for any $a \in \mathbb{R}$, $a \neq 0, 1, 2$

A SOLUTION.

One has $f = f_0 + f_1 + f_2$ where $f_0 = 1/2x$, $f_1 = -1/(x - 1)$ and $f_2 = 1/2(x - 2)$. It is easy to see that $r_{f_0}(a) = |a|$, $r_{f_1}(a) = |a - 1|$, $r_{f_2}(a) = |a - 2|$. Let $r(a) := \min(r_{f_0}(a), r_{f_1}(a), r_{f_2}(a))$

For any $a \in \mathbb{R}$, $a \neq 0, 1, 2$ the Taylor series $P_f(a, x)$ is equal to the sum $P_{f_0}(a, x) + P_{f_1}(a, x) + P_{f_2}(a, x)$. Since each of the series $P_{f_i}(a)(x)$, $1 \leq i \leq 3$ is convergent for $|x| < r(a)$ we see that $r_f(a) \leq r(a)$. I'll leave for you to check that $r_f(a) = r(a)$.

Let V be a vector space, $\mathcal{Q}(V)$ the space of quadratic functions on V and $\mathcal{B}_s^2(V)$ be the space of symmetric bilinear forms on V . To any symmetric bilinear form f on V we associate a function $q(f)$ on V where $q(f)(v) := f(v, v)$.

6. a) Show that for any $f \in \mathcal{B}_s^2(V)$ the function $q(f)$ on V is quadratic.

The part a) show that we have defined a map $\alpha : \mathcal{B}_s^2(V) \rightarrow \mathcal{Q}(V)$, $\alpha(f) := q(f)$.

b) Prove that α is an isomorphism between vector spaces $\mathcal{B}_s^2(V)$ and $\mathcal{Q}(V)$.

c) Construct explicitly a linear map $\beta : \mathcal{Q}(V) \rightarrow \mathcal{B}_s^2(V)$ which is the inverse of α .

A SOLUTION.

I'll construct a linear map $\beta : \mathcal{Q}(V) \rightarrow \mathcal{B}_s^2(V)$ and leave for you to check that β is the inverse of α . Of course this will give a proof of the part b).

For any $q \in \mathcal{Q}(V)$ we define $\beta(q) \in \mathcal{B}_s^2(V)$ by $\beta(q)(v, w) := 1/2[q(v + w) - q(v - w)]$.

d) Suggest a generalization of of the construction $f \rightarrow q(f)$ and try to generalize the problems a)-c).

7. Let f be a twice differentiable function on \mathbb{R}^2 and $g : \mathbb{R}^2 - \{x, 0\} \rightarrow \mathbb{R}^2$ be given by $g(x, y) := (xy, x/y)$.

Find first and second partial derivatives of the function $F := f \circ g$ in terms of the partial derivatives of the function f