MATH 23a, FALL 2002
THEORETICAL LINEAR ALGEBRA AND MULTIVARIABLE CALCULUS
Midterm (take-home portion)
October 30, 2002

Directions: You have until 11 A. M. on Monday, November 4, to complete this exam, when it will be collected in class. You may use your own class notes, your own homework assignments, the posted solution sets, and the course textbooks as your only aids. You may not use any internet resources except for the course website and the posted homework solutions. You may not discuss the exam with anyone, and all questions should be directed only to the instructor. (In particular, please do not direct questions to the Math 23a CA's.) Please note that I will hold office hours on Thursday and Friday afternoon as usual. I will check e-mail on Saturday afternoon (but not Sunday), so you should read the problems and ask all questions before then!

There is partial credit, but only for intelligible work. There are only two questions, and a significant part of the assignment is to write carefully thought-out solutions. Please write neatly, and please turn in clean copies of solutions (no scratch work, please!). In fact, one point per problem will be awarded for *neatness only*, and one point will be awarded for *style only*. Make sure your name is prominently displayed on your work, and *please* staple your final pages together into one stack.

You may quote results from class and/or your notes with an appropriate reference, and you must cite anything you take one of the class texts. Otherwise, all work should be your own.

1. Prove the **Second Isomorphism Theorem**, which states:

   If V and W are subspaces of a vector space U, then

   $$V/(V \cap W) \cong (V + W)/W.$$

2. Consider the field $F = \mathbb{Z}/2\mathbb{Z}$ with its elements identified as 0 and 1. (Properly speaking, these are representatives of equivalence classes, but we will allow the simpler notation.)

   Now define $F[x] = \{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n | n \in \mathbb{N}, a_i \in F, \forall i\}$ to be the vector space of all polynomials with coefficients in $F = \mathbb{Z}/2\mathbb{Z}$, where addition and scalar multiplication are defined as usual. Note, however, there is also already a notion of the multiplication of two polynomials, and that $F[x]$ actually satisfies the axioms for a ring.

   Let $p(x) = x^3 + x + 1$ be a fixed vector (polynomial) in $F[x]$, and define $I = \{a(x)p(x) | a(x) \in F[x]\}$ to be the subspace of $F[x]$ consisting of all (polynomial, not just scalar) multiples of this single vector.

   (a) Show that I is a subspace of vector space $F[x]$. (In fact, it is a *subring*, but we are only concerned with vector space properties in this part of the question.)

   (b) Define the quotient space $F[x]/I$ in terms of the data above, and find a minimal complete set of coset representatives. (Note that this is not the same as finding a basis. Since F is finite, it is possible to list all the elements of $F[x]/I$. Hint: You might consider long division of polynomials to help you classify the cosets.)

   (c) We have already seen in general that the quotient space has the structure of a vector space (so that we already have addition and scalar multiplication). Show that the natural definition of multiplication is well-defined for elements of the quotient space $F[x]/I$.

   (d) With the multiplication from part (c) and the representatives from part (b), show that $F[x]/I$ satisfies Axiom M3 and M4 (multiplicative identity and inverses) for a field by explicitly naming the identity and all multiplicative inverses. (This shows that $F[x]/I$ is a field because Axioms M1, M2, and D are inherited from the stucture of $F[x]$.)