MATH 23a, FALL 2001
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Final Exam (in-class portion)
January 19, 2001

Directions: You have three hours for this exam, though I have designed it to take less than the full amount of time. No calculators, notes, books, etc. are allowed. Please answer on the pages provided–there is a blank page at the end for scratch work. Note that not all questions are equally weighted.

Problem	Points	Score
1	36	
2	24	
3	15	
4	10	
5	15	
6	10	
7	10	
Total	120	

1. True or False (36 points, 2 each)

T or **F** Every field has either a prime number of elements or infinitely many elements.

T or **F** The columns of an $n \times n$ matrix A are linearly independent if and only if A is invertible.

T or **F** If A and B are linear maps from a vector space V to itself and $AB = I$, then $B = A^{-1}$.

T or **F** If U and V are vector spaces over the same field F, then $(U \oplus V)/V \cong U$.

T or **F** If $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear transformation and $n > m$, then $Ker(L)$ is non-trivial.

T or **F** If V is a vector space over the field $\mathbb{Z}/p\mathbb{Z}$ for some prime p, then V is finite-dimensional.

T or **F** If $A : U \longrightarrow V$ and $B : V \longrightarrow W$ are both surjective linear maps, then $B \circ A : U \longrightarrow W$ is also surjective.

T or **F** If V is a vector space over $\mathbb{R}$ and $dim(V) = 3$, then the space of alternating 2-linear forms on V is one-dimensional.

T or **F** If $A, B \in M_n(\mathbb{R})$, then $\det(A+B) = \det(A)+\det(B)$.

T or **F** The union of any collection of closed sets is closed.

T or **F** If $\{C_n\}$ is a nested sequence of compact sets in $\mathbb{R}^n$, then $\bigcap C_n$ is compact and non-empty.

T or **F** Every metric space has an associated norm.

T or **F** If V is a subspace of an inner product space U over $\mathbb{R}$, then $V \cap V^{\perp} = \{\mathbf{0}\}$.

T or **F** If $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is continuous and $S \subset \mathbb{R}^m$ is disconnected, then $f^{-1}(S)$ is disconnected in $\mathbb{R}^n$.

T or **F** If $A \subset \mathbb{R}^n$ is open and $\mathbf{x} \in A$, then there is some $\varepsilon > 0$ such that $B_\varepsilon(\mathbf{x}) \subset A$.

T or **F** If $A \subset \mathbb{R}^n$ is a closed set and $\mathbf{x} \in A$, then for every $\varepsilon > 0$, the ball $B_\varepsilon(\mathbf{x})$ contains points outside of A.

T or **F** If V is an inner-product space (over $\mathbb{R}$) and $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for V, then $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ when $i \neq j$.

T or **F** The sequence $\{a_n\}$ given by $a_n = 6^n$ is a Cauchy sequence in $\mathbb{Q}_3$.

2. True or False with Justification (24 points, 4 each)

Decide whether the following statements are true or false and give a *brief* justification–write no more than one sentence! (For example, if a statement is true, you might cite a theorem or provide a one-line proof. If a statement is false, you might provide a counter-example.)

T or **F** $\mathbb{Z}$ is bounded as a subset of $\mathbb{Q}_p$.

T or **F** If $A : U \longrightarrow V$ is a linear transformation and $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors of A with distinct eigenvalues, then $\mathbf{u}$ and $\mathbf{v}$ are linearly independent.

T or **F** If V is a finite-dimensional vector space spanned by a set of n vectors, then $dim(V) = n$.

T or **F** If V is an inner-product space with basis $\mathfrak{B}$, then it is possible to find an orthonormal basis for V.

T or **F** If $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is continuous and $S \subset \mathbb{R}^n$ is open, then $f(S)$ is open in $\mathbb{R}^m$.

T or **F** Every bounded infinite subset of $\mathbb{R}^n$ has an accumulation point.

3. Examples (15 points, 5 each)

Provide examples of the following. Justification is not required as long as the examples are correct.

(a) A subset $S \subset \mathbb{R}^n$ that is neither open nor closed.

(b) A vector space V and a linear transformation $A : V \longrightarrow V$ that has infinitely many eigenvalues.

(c) A metric space S and a non-trivial subset A that is both open and closed.

4. Similar Matrices (10 points)

(a) Let $A, B \in M_n(F)$. Define what it means for A and B to be *similar*.

(b) Show that if A and B are similar, then $\det(A) = \det(B)$.

5. Compactness (15 points)

(a) Let $A \subset \mathbb{R}^n$. Give the "open cover" definition saying what it means for A to be *compact*.

(b) Show that if $A \subset \mathbb{R}^n$ is compact and $B \subset A$ is closed, then B is compact.

(c) State the Heine-Borel Theorem.

6. Boundary of a set (10 points)

(a) Let $A \subset \mathbb{R}^n$. Define ∂A, the *boundary* of A.

(b) Show that if A is open, then $\partial A \cap A = \emptyset$.

7. Inner Products (10 points)

(a) Let V be a vector space. Give the general definition for an *inner product* on V.

(b) Show that $\langle A, B \rangle = \mathrm{tr}(A^t B)$ defines an inner product on $M_n(F)$, where $\mathrm{tr}(C)$ is the *trace* of the matrix C.