If you don't understand anything about any of the solutions here, or if you spot mistakes, feel free to e-mail me.

Important notes.

1. If $\det(AB) = 1$, this does not mean $AB = I$.

2. If $\det(A) = 0$, this does not mean that a column of A is all zeroes.

2

Method One.
Lemma. If $\sigma \in S_n$, then $\operatorname{sgn}(\sigma) = \operatorname{sgn}\sigma^{-1}$.
Proof. We can write σ as a product of transpositions $\alpha_1\alpha_2\ldots\alpha_n$. Note, then, that $\sigma^{-1} = \alpha_n\alpha_{n-1}\ldots\alpha_1$, so σ and σ^{-1} must have the same sign. Alternately, note that $\operatorname{sgn}(\cdot)$ is a multiplicative function, i.e. $\operatorname{sgn}(\sigma\tau) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)$.
Proof that $\det(A) = \det(A^t)$. Let $A = \{\alpha_{ij}\}$. Then, $A^t = \{\alpha_{ji}\}$. We can write,

$$\det(A) = \sum_{\pi \in S_n} \operatorname{sgn}(\pi)\alpha_{1,\pi(1)}\alpha_{2,\pi(2)} \cdots \alpha_{n,\pi(n)}$$

We can re-order the terms of the product inside the summation,

$$= \sum_{\pi \in S_n} \operatorname{sgn}(\pi)\alpha_{\pi^{-1}(1),1}\alpha_{\pi^{-1}(2),2} \cdots \alpha_{\pi^{-1}(n),n}$$

Using the lemma,

$$= \sum_{\pi \in S_n} \operatorname{sgn}(\pi^{-1})\alpha_{\pi^{-1}(1),1}\alpha_{\pi^{-1}(2),2} \cdots \alpha_{\pi^{-1}(n),n}$$

Iterating over π and π^{-1} is the same thing, so we can re-arrange the finite sum and substitute notation to get,

$$= \sum_{\pi \in S_n} \operatorname{sgn}(\pi)\alpha_{\pi(1),1}\alpha_{\pi(2),2} \cdots \alpha_{\pi^{-1}(n),n}$$
$$= \det(A^t).$$

Method 2 (outline). It's possible also to induct on n, the size of the matrix. The 1×1 case is clear, and then you can reduce the $(n+1) \times (n+1)$ case using expansion by minors, once in a row, and the second time in a column. It's easy to check that the coefficient of the minor $((-1)^{i+j})$ is the right one.

Method 3 (outline). Look at the way Curtis defines the determinant. Copy that, but re-define in terms of columns. Show that your new function satisfies all the properties of the determinant, and then declare it to be the determinant by the uniqueness of determinants. *Note: copying a proof out of homework is probably not the "intended" way for you to do the homework. It's always better (not always possible) to work from the definitions you get in class.*

4

Method One:
($\Rightarrow$): Since A is invertible, we have A^{-1}. Then,

$$\det(A)\det(A^{-1}) = \det(AA^{-1}) = \det(I) = 1,$$

so $\det(A) \neq 0$.

($\Leftarrow$): If $\det(A^{-1}) \neq 0$, then, since the determinant is an alternating bi-linear form, the set $S = \{Ae_1, Ae_2, \ldots, Ae_n\}$ is linearly independent. Since we're in an n-dimensional space, span $\{S\} = V$. Furthermore, since $\Im(A) = $ span $\{S\}$, A is surjective, which implies, because we're in a finite dimensional space, A is bijective.

Method Two: (Due to Michael Hoffman)

$$\det(A) \neq 0 \Leftrightarrow \det(A - \lambda I) \neq 0, \text{ where } \lambda = 0$$
$$\Leftrightarrow 0 \text{ is not an eigenvalue of } A$$
$$\Leftrightarrow \ker(A) = \left\{\vec{0}\right\}$$
$$\Leftrightarrow A \text{ is invertible.}$$

I liked this method because it reminded us of the connection we used between determinants and eigenvalues.