MATH 23A SOLUTION SET #1 (PART A)

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Problem 1. *Use mathematical induction to show that* $\sum_{i=1}^{k}(2i-1) = k^2$.

Proof. Let $C(k)$ be the given claim, that $\sum_{i=1}^{k}(2i-1) = k^2$. Proof by induction is done in the following two steps:

- **Inductive base** - Checking $C(1)$ is true.
- **Inductive step** - Showing that, for any $k \in \mathbb{N}$, $C(k) \Rightarrow C(k+1)$, or, in other words, assuming $C(k)$ is true , proving it follows that $C(k+1)$ is true. The assumption that $C(k)$ is true is called **Inductive hypothesis**.

Notice that if we define a set $P = \{k | C(k) \text{ is true}\} \subseteq \mathbb{N}$, then what we are doing above is showing that $1 \in P$ and $\forall k \in P$, $k+1 \in P$, so that by one of Peano's axioms, $P = \mathbb{N}$, and hence $C(k)$ is true for all $k \in \mathbb{N}$. When writing up a proof by induction, do not forget to check the base case, and always formulate what exactly your inductive hypothesis is - doing this using plain English ("assume", "want to show" etc.) might not be a bad idea! So here's how it might look:

When $k = 1$, we check that $\sum_{i=1}^{1}(2i-1) = 2 \cdot 1 - 1 = 1^2$

Now, assume that $\sum_{i=1}^{k}(2i-1) = k^2$ for some $k \in \mathbb{N}$. We want to show that $\sum_{i=1}^{k+1}(2i-1) = (k+1)^2$. Using our inductive hypothesis, we get that:

$$\begin{aligned} \sum_{i=1}^{k+1}(2i-1) &= \sum_{i=1}^{k}(2i-1) + 2(k+1) - 1 \\ &= k^2 + 2k + 1 \text{ (by inductive hypothesis)} \\ &= (k+1)^2 \end{aligned}$$

which proves the inductive step. □

Problem 2. *Verify that multiplication is well-defined for integers, as defined by ordered pairs of natural numbers.*

Proof. In class, we have defined $\mathbb{Z} = \{(a,b) | a, b \in \mathbb{N}\}/\sim$, where $(a,b) \sim (a',b')$ whenever $a + b' = a' + b$, and then multiplication on the equivalence classes was defined as $[(a,b)][(c,d)] = [(ac+bd, ad+bc)]$.

We want to show this definition does not depend on which representative of an equivalence class we choose when multiplying. So let $a, b, c, d, a', b' \in \mathbb{N}$ be such that $(a,b) \sim (a',b')$. Then, we have that:

$[(a,b)][(c,d)] = [(ac+bd, ad+bc)]$ and $[(a',b')][(c,d)] = [(a'c+b'd, a'd+b'c)]$

and we want to check that

$$[(ac+bd, ad+bc)] = [(a'c+b'd, a'd+b'c)]$$

or put differently, that

$$(ac+bd, ad+bc) \sim (a'c+b'd, a'd+b'c)$$

which amounts to showing that

$$ac+bd+a'd+b'c = ad+bc+a'c+b'd$$

Once you see how to group terms in the above expression, the rest is not hard to write up:

$$\begin{aligned}
& (a,b) \sim (a',b') \\
\Rightarrow\ & a+b' = b+a' \\
\Rightarrow\ & (a+b')c = (b+a')c \text{ and } (a+b')d = (b+a')d \\
\Rightarrow\ & (a+b')c + (b+a')d = (b+a')c + (a+b')d \\
\Rightarrow\ & ac+b'c+bd+a'd = bc+a'c+ad+b'd \text{ (by distributivity in } \mathbb{N}) \\
\Rightarrow\ & ac+bd+a'd+b'c = ad+bc+a'c+b'd \text{ (by commutativity of addition in } \mathbb{N}) \\
\Rightarrow\ & (ac+bd, ad+bc) \sim (a'c+b'd, a'd+b'c) \\
\Rightarrow\ & [(ac+bd, ad+bc)] = [(a'c+b'd, a'd+b'c)] \\
\Rightarrow\ & [(a,b)][(c,d)] = [(a',b')][(c,d)]
\end{aligned}$$

In a completely analogous way we can show that $\forall a,b,c,d,c',d' \in \mathbb{N}$ such that $(c,d) \sim (c',d')$ we have that

$$[(a,b)][(c,d)] = [(a,b)][(c',d')]$$

We cannot say, however, the above line holds because of commutativity, as many did on the homework, since we cannot claim multiplication in $\mathbb{Z}$ is commutative in order to show it can be defined in the first place! Notice also that the above proof does not use "minus" anywhere - another common mistake on this problem was to write things like $(a-b)c = ac-bc$ within the proof - but $a-b$ is an integer, so we don't know how to multiply it with anything before we finish up this problem!

Finally, multiplication in $\mathbb{Z}$ is well-defined since, $\forall a,b,c,d,a',b',c',d' \in \mathbb{N}$ such that $(a,b) \sim (a',b')$ and $(c,d) \sim (c',d')$ we have:

$$[(a,b)][(c,d)] = [(a,b)][(c',d')] = [(a',b')][(c',d')]$$

$\square$