5

First, we note that $\mathbb{Z}/2\mathbb{Z}$ contains two distinct elements, $\bar{0}$ and $\bar{1}$. Also, $\bar{1} = -\bar{1}$.

Claim: $\mathbb{Z}/2\mathbb{Z}$ is not an ordered field.
Proof (by contradiction): Assume $\mathbb{Z}/2\mathbb{Z}$ is an ordered field. Then there exists a set $P \subset \mathbb{Z}/2\mathbb{Z}$ satisfying P1-P3. For any set P, either $\bar{1} \in P$ or $\bar{1} \notin P$. If $\bar{1} \in P$, then $-\bar{1} = \bar{1} \in P$ and P1 fails. If $\bar{1} \notin P$, then $-\bar{1} \notin P$ and $-\bar{1} \neq \bar{0}$. Again, P1 fails. We've reached a contradiction, so we may conclude that $\mathbb{Z}/2\mathbb{Z}$ is not an ordered field.

Notes: Many of you did this problem by enumerating the possibilities for P. These were $\varnothing$, $\{\bar{1}\}$, $\{\bar{0}\}$, $\{\bar{0}, \bar{1}\}$. Note that $\varnothing = \{\}$ is the empty set, NOT $\{\varnothing\}$, which is the set that contains the empty set. P1 fails for $\varnothing$ not because $\varnothing$ is empty, but because we may choose $x = \bar{1}$, so that $\bar{1} \notin P$ and $-\bar{1} \notin P$ and $\bar{1} \neq \bar{0}$.

6

Claim: Additive inverses in a field are unique.
Proof: Let b and b' be inverses of a. Then $a + b = 0$ and $a + b' = 0$. Now,

$$
\begin{aligned}
b &= b + 0 && \text{(by def'n of idenity)} \\
&= b + (a + b') \\
&= (b + a) + b' && \text{(by associativity)} \\
&= 0 + b' \\
&= b'
\end{aligned}
$$

We've shown that for any two inverses b, b' of a, $b = b'$, so the additive inverse must be unique.