2

Addition

Recall that $(a, b) \sim (a', b')$ if $ab' = b'a$. We wish to show that the addition operation determined by $[(a, b)] + [(c, d)] = [(ac, bc + ad)]$ is well-defined. To do this, we need to show that it doesn't matter which represenative of the equivalence classes we choose to add.

Suppose $(a, b) \sim (a', b')$ and $(c, d) \sim (c', d')$. Then, $ab' = a'b$ and $cd' = c'd$. We can write,

$$\begin{aligned}
& (ab')cc' + aa'(cd') = ab'cc' + aa'cd' && \text{(identity)} \\
\implies & (a'b)cc' + aa'(c'd) = abcc' + aa'cd' && \text{(substitution from above)} \\
\implies & (a'c')(bc + ad) = (ac)(b'c' + a'd') && \text{(commutativity, distributivity, and associativity)} \\
\implies & (ac, ad + bc) \sim (a'c', b'c' + a'd') && \text{(definition of equivalence)} \\
\implies & [(a, b)] + [(c, d)] = [(a', b')] + [(c', d')].
\end{aligned}$$

Since (a', b') is an arbitrary element of $[(a, b)]$ and (c', d') is an arbitrary element of $[(c, d)]$, the addition operation does not depend on the representative of the equivalence class, and is therefore well-defined.

Multiplication Now we wish to show that the multiplication operation determined by $[(a, b)] \cdot [(c, d)] = [(ab, cd)]$ is well-defined.

Again, suppose $(a, b) \sim (a', b')$ and $(c, d) \sim (c', d')$. Then, $ab' = a'b$ and $cd' = c'd$. Now,

$$\begin{aligned}
& (ab')(cd') = ab'cd' && \text{(identity)} \\
\implies & (a'b)(c'd) = (ab')(cd') && \text{(substitution)} \\
\implies & (bd)(a'c') = (ac)(b'd') && \text{(commutativity and associativity)} \\
\implies & (ac, a'c') \sim (bd, b'd') && \text{(defn. of equivalence).}
\end{aligned}$$

Thus, $[(a, b)] \cdot [(c, d)] = [(a', b') \cdot (c', d')]$ which shows that multiplication does not depend on the representative of the equivalence class we choose. Therefore, it is well-defined.

Notes:

- Many of you did this proof upside-down. You started with $[(a, b)] + [(c, d)] = [(a', b') + (c, d)]$ and concluded that $(a, b) \sim (a'b')$. You can't do this. You want to prove: "If $(a, b) \sim (a', b')$, then $[(a, b)] + [(c, d)] = [(a', b')] + [(c, d)]$." But, you're trying to prove it by saying $[(a, b)] + [(c, d)] = [(a', b')] + [(c, d)] \implies \ldots \implies (a, b) \sim (a', b')$ or $0 = 0$. That's not valid. It's especially confusing because to work out the algebra, it's easiest to in fact work through the proof that way, but you have to write it up in the valid direction. Look at Isidora's solution set for Problem Set 1, Section A for another example of how to write this sort of proof. If this sort of proof is still unclear, stop by any CA's office hours, and we'll try to clear it up.

  In some cases, especially when dealing with algebraic equalities, a proof is "reversible," that is, the relation between every pair of consecutive steps is implication in both directions. In that case, write "if and only if" or use $\iff$. Reserve these reversible arguments for when you are trying to prove if and only if statements, and even then, it's often much clearer to prove the two directions seperately.

- If you're trying to prove statement X, you should never write "Assume X" at the top of your proof. If you assume your result and then prove some truth (e.g., $0 = 0$) you haven't proved anything except that your result could be true. If you want to write a proof by contradiction, you would write "Assume not X" and show that some contradiction follows.

- Many of you used division in the above algebraic arguments, or wrote $(a, b) \sim (a', b') \implies \frac{b}{a} = \frac{b'}{a'}$. Remember that $a, b \in \mathbb{Z}$, and we don't have multiplicative inverses ($\frac{1}{a}, a \neq 1$) in the integers. So you have to stick with addition and multiplication for this argument, because that's what we defined for the integers.

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Proof 1 of the Triangle Inequality

First, notice that $-|x| \leq x \leq |x|, x \in \mathbb{R}$: if $x \in P$, then $-|x| = -x \leq x = |x|$. Otherwise, $-|x| = x \leq -x = |x|$.

Now, take any $x, y \in \mathbb{R}$. We have

$$\begin{aligned} -|x| &\leq x \leq |x| \\ -|y| &\leq y \leq |y| \\ -|x| - |y| &\leq x + y \leq |x| + |y| && \text{(adding the inequalities).} \end{aligned}$$

From the definition of absolute value, this implies

$$|x + y| \leq |x| + |y|.$$

Proof 2 of the Triangle Inequality

This is my favorite proof of this result. Note that $x^2 \geq 0$ for all $x \in \mathbb{R}$. Let $a, b \in Reals$.

$$\begin{aligned} |a + b|^2 &= (a + b)^2 \\ &\leq a^2 + 2ab + b^2 \\ &\leq |a|^2 + 2|a||b| + |b|^2 \\ \implies |a + b|^2 &\leq (|a| + |b|)^2 \\ |a + b| &\leq |a| + |b|. \end{aligned}$$

I've done both proofs with the assumption that $a, b \in \mathbb{R}$. Since $\mathbb{Q}$ is a subfield of $\mathbb{R}$, the proof holds for $a, b \in \mathbb{Q}$.

Notes:

- Many of you went from $-|x| \leq x \leq |x|$ to the triangle inequality, saying $x = a + b$. But that only implies that $-|a + b| \leq a + b \leq |a + b|$, which isn't what you want to prove. Fitzpatrick makes the jump without explaining that he's added two inequalities; you have to supply the step he's missing.

- Many of you proved the triangle inequality by looking at cases. Though this method usually produces a perfectly valid proof, I encourage you to always look for a proof without resorting to cases.