MATH 23A SOLUTION SET #2B (PROBLEM 4)

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Claim 1. *There is a real number x that satisfies the equation $x^3 - 2$.*

Proof. Let $S = \{x | x \in \mathbb{R},\ x^3 < 2\} \subset \mathbb{R}$. Since $1 \in S$, S is nonempty. Since $x \in S \Rightarrow x^3 < 8$, we have that 2 is an upper bound for S. By the least upper bound property of $\mathbb{R}$, any nonempty subset of $\mathbb{R}$ bounded above has a least upper bound in $\mathbb{R}$, so S has a least upper bound, call it b, and this is usually denoted $b = \sup S$ where sup stands for *supremum*. Now, exactly one of the following three relations holds: $b^3 < 2$, $b^3 > 2$ or $b^3 = 2$.

Suppose first that $b^3 < 2$. We want to find an $\varepsilon > 0$ such that $(b + \varepsilon)^3 < 2$. Assuming $\varepsilon < 1$ we get:

$$(b+\varepsilon)^3 = b^3 + 3b^2\varepsilon + 3b\varepsilon^2 + \varepsilon^3 < 2 - (2 - b^3) + \varepsilon(3b^2 + 3b + 1)$$

so that:

$$\varepsilon < min\left\{1, \frac{2 - b^3}{3b^2 + 3b + 1}\right\} \Rightarrow (b+\varepsilon)^3 < 2$$

But for such an ε, $b + \varepsilon \in S$, and $b + \varepsilon > b$, so b is not an upper bound for S, contradiction!

Now suppose $b^3 > 2$. We want to find a $\varepsilon > 0$ such that $(b - \varepsilon)^3 > 2$.Assuming $\varepsilon < 1$ we get:

$$(b-\varepsilon)^3 = b^3 - 3b^2\varepsilon + 3b\varepsilon^2 - \varepsilon^3 > 2 + (b^3 - 2) - \varepsilon(3b^2 + 1)$$

so that:

$$\varepsilon < min\left\{1, \frac{b^3 - 2}{3b^2 + 1}\right\} \Rightarrow (b-\varepsilon)^3 > 2$$

For such a ε, $(b - \varepsilon)^3 > 2$ so that $b - \varepsilon > x$ for all $x \in S$. Hence, $b - \varepsilon$ is an upper bound for S. But $b - \varepsilon < b$, so that b is not the lowest upper bound. Contradiction!

We conclude that $b^3 = 2$, and b is the solution (in $\mathbb{R}$) to the equation $x^3 - 2 = 0$. □

Claim 2. *b is irrational.*

Proof. Suppose b is rational, and let $b = \frac{p}{q}$, where p, q are relatively prime, so that the fraction is in lowest terms. We now have that:

$$b^3 = \frac{p^3}{q^3} = 2 \ \Rightarrow\ p^3 = 2q^3 \ \Rightarrow p \ \text{ is even, } p = 2p_1$$

so that:

$$8p_1^3 = 2q^3 \ \Rightarrow\ 4p_1^3 = q^3 \ \Rightarrow\ q \text{ is even}$$

But now both p and q are even, which contradicts our assumption that they are relatively prime. We conclude that b is irrational. □