MATH 23a, FALL 2002
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
(Final Version) Homework Assignment # 3
Due: October 11, 2001

1. Read Sections 3–5 and 7 from Chapter 2 of Curtis.

2. (A) If V is a vector space and $\mathbf{v} \in V$, show that $(-1)\mathbf{v} = -\mathbf{v}$.

   (Note that there really is something to do here. The expression on the right is the additive inverse of the vector, and the one on the left is the vector multiplied by the scalar -1, and these are not *a priori* the same thing.)

3. (A) Show that the set of ordered triples (x, y, z) of real numbers satisfying the equations $2x + y = 0$ and $3y - z = 0$ form a vector space over the field $\mathbb{R}$.

4. (B) How many vectors are in the vector space $V = (\mathbb{Z}/p\mathbb{Z})^n$, where p is a fixed prime number? How many distinct one-dimensional subspaces does V have? How many distinct two-dimensional subspaces does V have?

5. (B) Recall the vector space $C[0, 1] = \{f : [0, 1] \to \mathbb{R} | f \text{ is continuous}\}$. Now, for each $c \in \mathbb{R}$, consider the set of vectors defined by

   $$V(c) = \{f : [0, 1] \to \mathbb{R} | f \text{ is continuous, and } f(0) = c\}.$$

   For which values of c is $V(c)$ a subspace of $C[0, 1]$? Explain.

6. (B) Find an infinite set of linearly independent vectors in $C[0, 1]$, not including any polynomials. (Our standards of proof will be much relaxed in this problem, but give *some* indication of why your set works.)

7. (C) Show that if V is a vector space with $dim(V) = n$, then any collection of $n + 1$ vectors in V is linearly dependent.

8. (C) Let S be some set with finite cardinality n. (For simplicity, you may assume that $S = \{1, 2, \ldots, n\}$.) Find a basis for the vector space of functions $V = \{f : S \to \mathbb{R}\}$, where addition and scalar multiplication are defined as for functions of one real variable.

9. (D) Let V be a vector space over the field F defined as all infinite sequences of elements of F:

$$V = \{(a_0, a_1, a_2, \dots) | a_i \in F, \forall i \geq 0\}$$

For $k = 1, 2$, we define the two subspaces:

$$\ell^k = \left\{ (a_0, a_1, a_2, \dots) \middle| a_i \in F, \forall i \geq 0, \text{and } \sum_{n=0}^{\infty} |a_n|^k < \infty \right\}$$

(a) (*Not required*) Show/convince yourself that V is a vector space.

(b) Show that ℓ^2 is a subspace of V by showing that the addition of vectors and scalar multiplication are closed operations. That is, if $\mathbf{u}, \mathbf{v} \in \ell^2$ and $a \in \mathbb{R}$, then $\mathbf{u} + \mathbf{v} \in \ell^2$ and $a\mathbf{u} \in \ell^2$.

(c) Show that if $F = \mathbb{R}$, then $\ell^1 \subset \ell^2$, and that $\ell^1 \neq \ell^2$.

(d) (*Bonus*) Consider the spaces ℓ^1 and ℓ^2 defined above with the terms a_i taken from the finite field $F = \mathbb{Z}/2\mathbb{Z}$. Give an explicit condition for deciding whether of not a particular sequence is in ℓ^1 or ℓ^2. (What must happen for such an infinite series to converge? Recall that the limits and sums are being taken in the finite field!)

10. (E) Let $V = \{(a_0, a_1, a_2, \dots) | a_i \in \mathbb{R}\}$ be the vector space of all infinite sequences of real numbers. Let W be the subspace of V consisting of all *arithmetic* sequences. Find a basis for W, and determine the dimension of W. (*A sequence is arithmetic is there is some constant c such that $a_{n+1} - a_n = c$ for all $n \geq 0$.*)