MATH 23b, SPRING 2002
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Homework Assignment # 3
Due: March 1, 2002

Homework Assignment #3 (Final Version)

1. Read Edwards, Sections 2.4–2.5 and 3.2–3.3. In particular, we will be proving the Inverse Function Theorem and Implicit Function Theorem (though not as Edwards does) before using them to help classify extrema of functions.

2. Prove the lemma from class that preceded the Inverse Function Theorem. Recall that is states:

    Lemma. Let $A \subset \mathbb{R}^n$ be an open rectangle, and suppose that $f : A \longrightarrow \mathbb{R}^n$ is continuously differentiable. If there is some $M > 0$ such that $|\frac{\partial f_i}{\partial x_j}(\mathbf{x})| \leq M, \forall \mathbf{x} \in A$, then $||f(\mathbf{y}) - f(\mathbf{z})|| \leq n^2 M ||\mathbf{y} - \mathbf{z}||$, $\forall \mathbf{y}, \mathbf{z} \in A$.

    (Hints: Use the Mean-Value Theorem on the coordinate functions of f. You might start by expanding $f_i(\mathbf{y}) - f_i(\mathbf{z})$ in terms of the coordinates of the domain vectors in a clever way.)

3. Edwards p. 194, problem #3.1.

    Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be given by $f(x, y) = (\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2})$. Show that f is locally invertible in a neighborhood of every point except the origin, and compute f^{-1} explicitly.

4. Consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by:

$$f(x) = \begin{cases} \frac{x}{2} + x^2 \sin(\frac{1}{x}) & , \quad \text{if } x \neq 0 \\ 0 & , \quad \text{if } x = 0 \end{cases}$$

   (a) Show that f is differentiable at 0 and that $f'(0) = \frac{1}{2}$.

   (b) Show that there is no open set (interval) containing 0 on which f is one-to-one.

5. Related to Edwards p. 194, problem #3.2(b).

   (a) Suppose that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is locally invertible at every point of its domain (that is, $f'(a) \neq 0, \forall a \in \mathbb{R}$). Show that f is one-to-one.

   (b) Consider $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by $g(x, y) = (e^x \cos y, e^x \sin y)$. Show that g is locally invertible at every point of its domain (that is, $\det[Jg(\mathbf{x})] \neq 0, \forall \mathbf{x} \in \mathbb{R}^2$), but that g is not one-to-one.

6. Consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $f(x, y) = x^2 + y^3 + e^y$, and let C be the curve defined by the level set $f(x, y) = 0$.

   (a) Show that there is a neighborhood of every point of C in which y may be defined implicitly as a function of x.

   (b) Find $\frac{dy}{dx}$ at every point of C. (*Hint: Consider the result of problem # 1.4 on p. 171 of Edwards.*)

7. An $n \times n$ matrix A is said to be in *block form* when we choose integers $1 = n_0 \leq n_1 \leq \ldots \leq n_k = n$ and write

$$
A = \begin{bmatrix} A_{11} & \cdots & A_{1k} \\ \vdots & & \vdots \\ A_{k1} & \cdots & A_{kk} \end{bmatrix},
$$

where A_{ij} is an $(n_i - n_{i-1}) \times (n_j - n_{j-1})$ matrix of the corresponding entries of A. (In particular, note that A_{ii} is square, $\forall i$.)

   (a) In the special case where A is $2n \times 2n$ and we write A as four $n \times n$ blocks,

$$
A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},
$$

   show by example that, in general,

$$
\det(A) \neq \det(A_{11}) \det(A_{22}) - \det(A_{12}) \det(A_{21}).
$$

   (b) In the same special case as part (a), show that

$$
\det(A) = det(A_{11}A_{22} - A_{12}A_{21})
$$

   provided that all of the blocks A_{ij} commute pairwise under multiplication. (This result is known as Schur's Formula.)

   (c) In the special case where A is $(n + m) \times (n + m)$ and we write A with two diagonal blocks of size $n \times n$ and $m \times m$, A is said to be *block upper triangular* if $A_{21} = 0$:

$$
A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},
$$

   Show that $\det(A) = \det(A_{11}) \det(A_{22})$ in this case.

8. (Not required) Problem #3.5 from Edwards, p. 194. Show that the equations $\sin(x + z) + \log yz^2 = 0$ and $e^{x+z} + yz = 0$ define z implicitly as a function of x and y near the point $(1, 1, -1)$.