Solution Set 2

Math 23a
October 14, 2002

2. Let $\mathbf{v}$ be an element of a vector space V. We have

$$\begin{aligned} \mathbf{0} &= (1-1)\cdot\mathbf{v} && \text{since } 0\cdot\mathbf{v}=\mathbf{0} \\ &= 1\cdot\mathbf{v} + (-1)\cdot\mathbf{v} && \text{by distributivity} \\ &= \mathbf{v} + (-1)\cdot\mathbf{v} && \text{since } 1\cdot\mathbf{v}=\mathbf{v} \end{aligned}$$

so since additive inverses are unique, $(-1)\cdot\mathbf{v} = (-\mathbf{v})$.

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3. Let

$$V = \{(x,y,z)\in\mathbf{R}^3 \mid 2x+y=0 \text{ and } 3y-z=0\}.$$

We claim that V is a vector space. Since $V\subset\mathbf{R}^3$ by definition, we only have to show that V is a vector subspace of $\mathbf{R}^3$, which involves proving the following:

- The set V is nonempty: clearly $(0,0,0)\in V$ since $2\cdot 0+0 = 3\cdot 0 - 0 = 0$.

- The set V is closed under addition: suppose

  $$\mathbf{v}_1 = (x_1,y_1,z_1)\in V \quad\text{and}\quad \mathbf{v}_2=(x_2,y_2,z_2)\in V.$$

  Then $\mathbf{v}_1+\mathbf{v}_2 = (x_1+x_2, y_1+y_2, z_1+z_2)$, and

  $$2(x_1+x_2)+(y_1+y_2) = (2x_1+y_1)+(2x_2+y_2) = 0+0 = 0$$

  and

  $$3(y_1+y_2)-(z_1+z_2) = (3y_1-z_1)+(3y_2-z_2) = 0+0=0$$

  so $\mathbf{v}_1+\mathbf{v}_2\in V$.

- The set V is closed under multiplication: suppose $\mathbf{v}=(x,y,z)\in V$ and $c\in\mathbf{R}$. Then $c\cdot\mathbf{v} = (c\cdot x, c\cdot y, c\cdot z)$, and

  $$2(c\cdot x)+c\cdot y = c\cdot(2x+y) = c\cdot 0 = 0$$

  and

  $$3(c\cdot y) - c\cdot z = c\cdot(3y-z) = c\cdot 0 = 0$$

  so $c\cdot\mathbf{v}\in V$.

Notes on these problems:

(1) The set

$$\{(x, y, z) \in \mathbf{R}^3 \mid 2x + y = 0 \ \text{ and } \ 3y - z = 0\}$$

is contained in, but is *not* the same as, the set

$$\{(x, y, z) \in \mathbf{R}^3 \mid 2x - 2y + z = 0\}.$$

Consider $x = y = 1$ and $z = 0$: then $2x - 2y + z = 0$ but $2x + y = 3y - z = 3$. This comes from the fact that $a - b = 0$ does not imply that $a = b = 0$.

(2) You were not allowed to assume that a vector $\mathbf{v} \in V$ can be written in coordinates $\mathbf{v} = (a_1, \ldots, a_n)$ or even with an infinite number of coordinates $\mathbf{v} = (a_1, a_2, \ldots)$. You had to consider a completely abstract vector space V. It is true that every finite-dimensional vector space is isomorphic to F^n for some n, but the proof of that uses the fact that $(-1) \cdot \mathbf{v} = -\mathbf{v}$ anyway. And it is not at all clear that all infinite-dimensional vector spaces have a basis; you need the Axiom of Choice for that proof (which we have not done). And even if you believe the Axiom of Choice, a vector space may have an uncountable number of basis elements, in which case you can't write a vector as $(a_1, a_2, \ldots)$ in the first place.

In pure math you should avoid choosing a basis if at all possible, because everything almost always works out much more nicely that way. Choosing a basis is like wimping out (unless it's totally unavoidable). Trust me on this. More technically: choosing a basis forfeits the *naturality* of your proofs (*naturality* is actually a category theory term!) and makes it much more messy to show that things you conclude about your vector spaces will carry over in a nice way when you do things like take quotients and direct sums and tensor products.

(3) Recall from class that the three properties I demonstrated for the set V in problem 3 really are the only things I needed to verify. The reason for this is that all of the vector space axioms carry over from the fact that $\mathbf{R}^3 \supset V$ is a vector space. But if you did show that V satisfied all eight axioms, *including* the closure properties (i.e. "there exists maps $+ : V \times V \to V$ and $\cdot : F \times V \to V$") then of course you got full credit, and you did a valuable exercise that everyone should do at least once in their life.

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