

MATH 23b, SPRING 2002  
THEORETICAL LINEAR ALGEBRA  
AND MULTIVARIABLE CALCULUS  
Homework Assignment # 4  
Due: March 8, 2002

Homework Assignment #4 (Final Version)

1. Read Edwards:
  - Section 2.4 (especially Example 3, Theorems 4.3 and 4.4 from pp. 94–97)
  - Section 2.6 (as review only!)
  - Section 2.7 (intro through Theorem 7.1 from pp. 129–131 and the discussion through Example 6 on pp. 137–140)
  - Section 2.8.

2. From Edwards p. 194, problem #3.7:

Consider the set  $S$  of points in  $\mathbb{R}^5$  defined by the two equations:

$$xu^2 + yzv + x^2z = 3$$

$$xyv^3 + 2zu - u^2v^2 = 2$$

Show that there is a neighborhood of the point  $(1, 1, 1, 1, 1) \in S$  and a differentiable function  $h : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  such that  $h(x, y, z) = (u, v)$  on the neighborhood, and find  $Jh(1, 1, 1)$ .

3. Let  $\mathbf{v}_1, \dots, \mathbf{v}_m$  be linearly independent vectors in  $\mathbb{R}^n$ .

Let  $V = \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ , and let  $V^\perp$  be the orthogonal complement to  $V$ . Show that  $\dim(V^\perp) = n - m$ .

4. Taken from Edwards p. 116, problem #5.9:

Given  $x_1, \dots, x_n$  be positive real numbers, we define their arithmetic and geometric means as follows:

$$A.M. = \frac{x_1 + \dots + x_n}{n}$$

$$G.M. = \sqrt[n]{x_1 \cdots x_n}$$

Use Lagrange multipliers to prove that the geometric mean is always less than or equal to the arithmetic mean by minimizing the function  $f(x_1, \dots, x_n) = \frac{1}{n}(x_1 + \dots + x_n)$  on the set  $S = \{\mathbf{x} | g(\mathbf{x}) = 0\}$ , where  $g(x_1, \dots, x_n) = x_1 \cdots x_n - 1$ .

5. Edwards p. 99, problem #4.13. If a triangle has side lengths  $x$ ,  $y$ , and  $z$  and perimeter  $2s = x + y + z$ , then its area  $A$  is given by  $A^2 = s(s-x)(s-y)(s-z)$ . Show that, among triangles with a given perimeter, the one with the largest area is equilateral. (*Bonus: Prove the area formula!*)
6. Edwards p. 116, problem #5.12.. Find the points on the line  $x + y = 10$  and the ellipse  $x^2 + 2y^2 = 1$  which are closest.
7. Adapted from Edwards p. 141, problem #7.4:

Let  $f(x, y, z) = xy^2z^3$ , and consider the point  $\mathbf{a} = (1, 0, -1)$ . Find the second-order Taylor polynomial  $T_2$  for  $f$  at  $\mathbf{a}$ , and show directly that the second-order remainder, defined as  $R_2(\mathbf{h}) = f(\mathbf{a} + \mathbf{h}) - T_2(\mathbf{h})$ , satisfies:

$$\lim_{\|\mathbf{h}\| \rightarrow 0} \frac{R_2(\mathbf{h})}{\|\mathbf{h}\|^2} = 0$$

Alternatively, if we think of  $T_2$  as a function of  $(x, y, z)$ , we can define  $R_2(x, y, z) = f(x, y, z) - T_2(x, y, z)$  and show that:

$$\lim_{(x,y,z) \rightarrow (1,0,-1)} \left( \frac{R_2(x, y, z)}{(x-1)^2 + y^2 + (z+1)^2} \right) = 0$$

8. (Not required) Let  $f(x, y) = x^2 \sin y$ , and consider the point  $\mathbf{a} = (3, \frac{\pi}{2})$ . Find the  $n$ -th order Taylor polynomials  $T_n$  for  $f$  at  $\mathbf{a}$  when  $n = 0, 1, 2, 3$ , and express the  $T_n(x, y)$  as polynomials in  $(x - 3)$  and  $(y - \frac{\pi}{2})$ . How does  $T_3$  compare with the third-order Taylor polynomials for  $g(x) = x^2$  at  $a = 3$  and for  $h(y) = \sin y$  at  $b = \frac{\pi}{2}$ ? Can you predict  $T_4$  for  $f$  at  $\mathbf{a}$  in terms of the fourth-order Taylor polynomials for  $g$  and  $h$  (at  $a$  and  $b$ , respectively)?