Math 23a, 2002.
Solution Set 4, Question 2.
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Question 2. Let $\mathbf{u} = (1, 0, 6)$, $\mathbf{v} = (1, 2, 1)$, and $\mathbf{w} = (2, 1, 3)$ be three vectors in F^3, that is, the set of ordered triples with coordinates in F. Find coefficients a, b, $c \in F$ to express the vector $\mathbf{x} = (1, 2, 3)$ as a linear combination of $\mathbf{x} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$.

Answer. This problem asks us to find a, b, c such that

$$a\begin{pmatrix}1\\0\\6\end{pmatrix} + b\begin{pmatrix}1\\2\\1\end{pmatrix} + c\begin{pmatrix}2\\1\\3\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix}.$$

This is really the same thing as finding the solutions to the system of linear equations:

$$\begin{array}{rcl} a+b+2c & = & 1 \\ 2b+\ c & = & 2 \\ 6a+b+3c & = & 3 \end{array}.$$

But only the coefficients make a difference. So we slap them into an augmented matrix and row reduce using the three elementary row operations while applying Gauss-Jordan elimination (see my section notes or Curtis or ask someone). We'll start with

$$\left[\begin{array}{ccccc} 1 & 1 & 2 & : & 1 \\ 0 & 2 & 1 & : & 2 \\ 6 & 1 & 3 & : & 3 \end{array}\right] \text{ and hope to end up with } \left[\begin{array}{ccccc} 1 & 0 & 0 & : & a \\ 0 & 1 & 0 & : & b \\ 0 & 0 & 1 & : & c \end{array}\right].$$

So let's go. I'll mark my row operations on the side and denote rows with Roman numerals.

$$\left[\begin{array}{ccccc} 1 & 1 & 2 & : & 1 \\ 0 & 2 & 1 & : & 2 \\ 6 & 1 & 3 & : & 3 \end{array}\right] \begin{array}{l} \\ \div 2 \\ \text{-6I} \end{array} \sim \left[\begin{array}{ccccc} 1 & 1 & 2 & : & 1 \\ 0 & 1 & 4 & : & 1 \\ 0 & 2 & 5 & : & 4 \end{array}\right] \begin{array}{l} \text{-II} \\ \\ \text{-2II}, \div 4 \end{array} \sim \left[\begin{array}{ccccc} 1 & 0 & 5 & : & 0 \\ 0 & 1 & 4 & : & 1 \\ 0 & 0 & 1 & : & 4 \end{array}\right] \begin{array}{l} \text{-5III} \\ \text{-4III} \\ \end{array} \sim$$

$$\left[\begin{array}{ccccc} 1 & 0 & 0 & : & 1 \\ 0 & 1 & 0 & : & 6 \\ 0 & 0 & 1 & : & 4 \end{array}\right].$$

Trade in our matrix for a system of equations to reveal that

$$\begin{array}{rcl} a & = & 1 \\ b & = & 6 \\ c & = & 4 \end{array}.$$

And just as a quick check, note that

$$\begin{pmatrix}1\\0\\6\end{pmatrix} + 6\begin{pmatrix}1\\2\\1\end{pmatrix} + 4\begin{pmatrix}2\\1\\3\end{pmatrix} = \begin{pmatrix}15\\16\\24\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix}.$$