If you don't understand anything about any of the solutions here, or if you spot mistakes, feel free to e-mail me at `zeyliger@fas.harvard.edu`. Homework problems have a tendency to creep up on exams, so be sure you know how to do all the assigned homework.

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a. **Claim:** $L : P^n \to \mathbb{R}$, defined by $Lp(x) = \int_0^1 p(x)\,dx$, is a linear operator.
Proof: We need to show that $cL(p(x)) = L(cp(x))$ and $L(p(x) + q(x)) = L(p(x) + L(q(x))$ for all $p(x), q(x) \in P^n$ and $c \in \mathbb{R}$.

$$\begin{aligned} cL(p(x)) &= c\int_0^1 p(x)\,dx \\ &= \int_0^1 cp(x)\,dx \\ &= L(cp(x)). \end{aligned}$$

$$\begin{aligned} L(p(x) + q(x)) &= \int_0^1 p(x) + q(x)\,dx \\ &= \int_0^1 p(x)\,dx + \int_0^1 q(x)\,dx \\ &= L(p(x)) + L(q(x)). \end{aligned}$$

■

Note: Many, many of you expanded $p(x)$ in terms of coefficients and computed the integrals directly. It is much simpler (and shorter) to just use the properties of the integral that you know from Calculus.

NEVER use the same name for two sets of coefficients. $p(x) = a_0 + a_1x + a_2x^2 + \ldots$ and $q(y) = a_0 + a_1y + a_2y^2 + \ldots$ are the same polynomial in our context, P^n! Polynomials are determined by their coefficients, and the a_i are the cofficients. Use a_i and b_i to to characterize two polynomials.

b. **Claim:** $\text{Im}(L) = \mathbb{R}$.
Proof: By definition, the image of L is a subset of the range of L, so $\text{Im}(L) \subset \mathbb{R}$. Consider $c \in \mathbb{R}$. Let $p(x) = c \in P^n$. $L(p(x)) = \int_0^1 c\,dx = c$, so $c \in \text{Im}(L)$. This implies that $\mathbb{R} \subset \text{Im}(L)$, and therefore $\text{Im}(L) = \mathbb{R}$. ■

Proof: (Even quicker) Note that $L(1) = 1$. Since 1 spans $\mathbb{R}$ and L is linear, $\text{Im}(L) \subset \text{span}\{L(1)\} = \text{span}\{1\} = \mathbb{R}$.

For any $x \in \mathbb{R}$, $x \cdot 1 = x$, so $\{1\}$ spans $\mathbb{R}$, and, since it is a one-element set, $\{1\}$ is a basis for $\mathbb{R}$. ■

Note: I took off a point if you took the integral but didn't explicitly show you you concluded that the image is the reals. To show two sets are equal, you need to show that every element of one is in the other. Or, equivalently, that each is a subset of the other.