MATH 23A SOLUTION SET #4 (PART D)

ISIDORA MILIN

Problem (2). *In this problem, we consider the shift operator. Consider the linear map* $S : \mathbb{R}^3 \to \mathbb{R}^3$ *which acts as follows:*

$$S(x, y, z) = (0, x, y)$$

Find the kernel and the image of S *and verify that*

$$dim(Ker(S)) + dim(Im(S)) = dim(\mathbb{R}^3)$$

Solution. Let's begin with $Ker(S)$.

$$\begin{aligned} & (x, y, z) \in Ker(S) \\ \Leftrightarrow\quad & S(x, y, z) = (0, x, y) = (0, 0, 0) \\ \Leftrightarrow\quad & x = y = 0 \end{aligned}$$

so that $Ker(S) = \{(x, y, z)|x, y, z \in \mathbb{R}, x = y = 0\}$. It is easily verified that $\{(0, 0, 1)\}$ spans $Ker(S)$, and it is clearly a linearly independent set, so it is a basis for $Ker(S)$ and $dim(Ker(S)) = 1$.

As for $Im(S)$, let $U = \{(x, y, z)|x, y, z \in \mathbb{R}, x = 0\}$. We show $Im(S) = U$ by showing that $(x, y, z) \in Im(S) \Leftrightarrow (x, y, z) \in U$, as follows:

$$\begin{aligned} & (x, y, z) \in Im(S) \\ \Leftrightarrow\quad & \exists (a, b, c) \in \mathbb{R}^3 \text{ s.t. } (x, y, z) = S(a, b, c) = (0, a, b) \\ \Leftrightarrow\quad & \exists (a, b, c) \in \mathbb{R}^3 \text{ s.t. } x = 0,\ y = a,\ z = b \\ \Leftrightarrow\quad & (x, y, z) \in U \end{aligned}$$

It is easily verified that $\{(0, 1, 0), (0, 0, 1)\}$ spans $Im(S)$, and also that it is linearly independent, so it is a basis for $Im(S)$ and $dim(Im(S)) = 2$.

It is left to check the dimension count:

$$dim(Ker(S)) + dim(Im(S)) = 1 + 2 = 3 = dim(\mathbb{R}^3)$$

□

Problem (3). *We generalize the notion of the shift operator. Let* V *be the vector spaces of all infinite sequences of real numbers, as in problem 3.9, and consider the linear maps* $S : V \to V$, $T : V \to V$, *where* S *and* T *act as follows:*

$$S(a_0, a_1, a_2, \ldots) = (0, a_0, a_1, a_2, \ldots)$$
$$T(a_0, a_1, a_2, \ldots) = (a_1, a_2, \ldots)$$

(1) *Find the kernel and image of* S. *How does the result about the dimensions of kernels and images apply?*

(2) *Show that* $T \circ S = I$*, but* $S \circ T \neq I$*, where* $I : V \to V$ *is the identity map.*

(3) *Which of* S *and* T *is onto? Which is one-to-one? Which is invertible? Explain.*

Solution. (1) To find $Ker(S)$:

$$\begin{aligned} & (a_0, a_1, \ldots) \in Ker(S) \\ \Leftrightarrow \quad & (0, a_0, a_1, \ldots) = S(a_0, a_1, \ldots) = (0, 0, 0, \ldots) \\ \Leftrightarrow \quad & a_i = 0 \text{ for all } i \in \mathbb{N} \end{aligned}$$

Thus, $Ker(S) = \{(0, 0, 0, \ldots)\}$ and $dim(Ker(S)) = 0$.

To find $Im(S)$:

$$\begin{aligned} & (a_0, a_1, \ldots) \in Im(S) \\ \Leftrightarrow \quad & \exists (b_0, b_1, \ldots) \in V \text{ s.t. } (a_0, a_1, \ldots) = S(b_0, b_1, \ldots) = (0, b_0, b_1, \ldots) \\ \Leftrightarrow \quad & a_0 = 0 \end{aligned}$$

Thus, $Im(S) = \{(a_0, a_1, \ldots) | a_0 = 0\}$. $Im(S)$ thus contains infinitely many linearly independent vectors in V - just consider the set of those which, for some $i \in \mathbb{N} - \{1\}$ have zero entries everywhere except at the i-th slot. Thus, $dim(Im(S)) = \infty$.

To interpret the dimension of kernels and images formula in this case, we could write something as follows:

$$dim(Ker(S)) + dim(Im(S)) = 0 + \infty = \infty = dim(V)$$

but still remember that this formula was proven in class only for finite-dimensional vector space V.

(2) To show $T \circ S = I$, it is enough to show that $T \circ S(v) = v$ for all $v \in V$, which is easy enough:

$$T \circ S(a_0, a_1, a_2 \ldots) = T(0, a_0, a_1, a_2, \ldots) = (a_0, a_1, a_2, \ldots)$$

To show $S \circ T \neq I$, it is enough to find just one vector $v_0 \in V$ for which $S \circ T(v_0) \neq v_0$. So consider $v_0 = (1, 0, 0, \ldots)$.

$$S \circ T(1, 0, 0, \ldots) = S(0, 0, \ldots) = (0, 0, 0, \ldots) \neq (1, 0, 0, \ldots)$$

(3) Recall that a linear map is one-to-one *iff* its kernel is the trivial vector space, and it is onto *iff* its image is whole of V. A map is invertible *iff* it is both one-to-one and onto.

S is not onto, since $Im(S) \neq V$ as we have seen in (1). S is one-to-one, since we have shown that $Ker(S) = \{(0, 0, 0, \ldots)\}$. Since S is not onto, it is not invertible.

We have also seen above that $v_0 = (1, 0, 0, \ldots)$ is such that $T(v_0) = (0, 0, \ldots)$, so that $v_0 \in Ker(T)$ and hence $Ker(T) \neq \{\vec{0}\}$ and T is not one-to-one. T is onto, however. To see that $Im(T) = V$, just note that any $v = (a_0, a_1, a_2, \ldots) \in V$ is in $Im(T)$. Namely $T \circ S = I$ implies that T maps $S(v)$ to v for all $v \in V$. Since T is not one-to-one, it is not invertible either. □