Solution Set 5

Math 23a
October 25, 2002

5. a) Suppose that A is surjective onto V and B is surjective onto W. We claim that $B \circ A : U \to W$ is surjective. Suppose that $w \in W$. By the surjectivity of B, we can find a $v \in V$ such that $Bv = w$, and by the surjectivity of A, we can find a $u \in U$ such that $Au = v$. Then we have

$$B \circ A(u) = Bv = w$$

so $w \in \operatorname{im}(B \circ A)$. Since $w \in W$ was arbitrary, $B \circ A$ is surjective.

b) Suppose that B is injective, so $\ker B = \{0\}$. We claim that $\ker(B \circ A) = \ker A$. First we show that $\ker A \subset \ker(B \circ A)$. Suppose that $u \in \ker A$, i.e. $Au = 0$. Then $B \circ A(u) = B(0) = 0$, so $u \in \ker(B \circ A)$ (note that this works whether or not B is injective). Now we show that $\ker(B \circ A) \subset \ker A$. Suppose that $u \in \ker(B \circ A)$, i.e. $B(Au) = 0$. Then since B is injective, $Au = 0$, so $u \in \ker A$. Thus $\ker(B \circ A) \subset \ker A$ and $\ker A \subset \ker(B \circ A)$, so the two must be equal (see note 1).

c) This is just an application of rank-nullity. We have

$$\begin{aligned}
\dim U &= \dim \operatorname{im} A + \dim \ker A = \dim V + \dim \ker A \\
\dim V &= \dim \operatorname{im} B + \dim \ker B = \dim W + \dim \ker B \\
\dim U &= \dim \operatorname{im}(B \circ A) + \dim \ker(B \circ A) = \dim W + \dim \ker(B \circ A)
\end{aligned}$$

where the last equation follows from part (a). Substituting $\dim U = \dim V + \dim \ker A$ and $\dim W = \dim V - \dim \ker B$ into this last equation, we have

$$\begin{aligned}
& \dim V + \dim \ker A = \dim V - \dim \ker B + \dim \ker(B \circ A) \\
\Longrightarrow\ & \dim \ker A + \dim \ker B = \dim \ker(B \circ A).
\end{aligned}$$

Notes on this problem:

(1) When you want to show that two sets A and B are equal, it is almost always the case that the best way to do this is to show that $A \subset B$ and $B \subset A$. Everything is much more clear this way.

(2) More notes will be forthcoming once I've graded the problem sets.

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