MATH 23a, FALL 2002
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
(Final Version) Homework Assignment # 6
Due: November 8, 2002

1. Read Sections 22–23 from Chapter 7 of Curtis.

2. (A) Suppose U is a finite-dimensional vector space and V is a subspace of U. Show that $dim(U/V) = dim(U) - dim(V)$.

3. (A) Let P_n be the vector space of polynomials (with real coefficients) of degree less than or equal to n, and let P_n^0 be the subspace of polynomials with terms of only even degree. Find a basis for P_n/P_n^0.

4. (B) Recall (HW #5.4) the linear differential operator $D : C^\infty \longrightarrow C^\infty$ given by $D(f) = f' + af$, where a is some fixed real number.

   Find *all* real eigenvalues and eigenvectors for this operator.

5. (B) Suppose λ is a non-zero eigenvalue for the linear transformation $A : V \longrightarrow V$.

   (a) Show that λ^2 is an eigenvalue for A^2.

   (b) If A is invertible, show that λ^{-1} is an eigenvalue for A^{-1}.

6. (C) Suppose $A : V \longrightarrow V$ is a linear map, and suppose $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is a set of non-zero eigenvectors for A with distinct eigenvalues $\lambda_1, \ldots, \lambda_m$. Show that these vectors are linearly independent. (Hint: Use induction on m.)

7. (C) Let $V = (\mathbb{Z}/7\mathbb{Z})^3$, and consider the linear map $L : V \longrightarrow V$ given by $L(x, y, z) = (x + y + z, 2y + 3z, 4z)$. Find the eigenvalues of L, and find an eigenbasis for V. (Hint: Look for likely choices of eigenvalues–I claim that two of them are easy, and the third follows a pattern.)

8. (D) Find examples of *invertible* linear transformations $A : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ such that:

   (a) A has no eigenvalues.

   (b) A has only one eigenvalue λ, but $dim(V_\lambda) < 4$.

   (c) $\mathbf{e}_1 = (1, 0, 0, 0)$ and $\mathbf{v} = (1, 1, 1, 1)$ are both eigenvectors but have distinct eigenvalues.

9. (D) Two linear transformations $A : V \longrightarrow V$ and $B : V \longrightarrow V$ are said to be *similar* if there exists an invertible linear transformation $S : V \longrightarrow V$ such that $A = SBS^{-1}$. Consider the following, where a well-written answer may suffice for both parts:

   (a) Show that if A and B are similar, then they have the same spectra. (That is, if λ is an eigenvalue for A, then λ is an eigenvalue for B, and vice versa.)

   (b) Suppose A and B are similar and λ is an eigenvalue for both. Find the precise relationship between the eigenspace for λ with respect to A and the eigenspace for λ with respect to B.