

MATH 23b, SPRING 2002
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
Homework Assignment # 6
Due: April 5, 2002

Homework Assignment #6 (Final Version)

1. Read Edwards Sections 4.2 and 4.3.
2. In class, we proved the following theorem:

Theorem. Let $A \subset \mathbb{R}^n$ be a closed rectangle, and let $f : A \rightarrow \mathbb{R}$ be a bounded function. If f is continuous at a , then $o(f, a) = 0$.

Prove the converse. (Recall that $o(f, a)$ is the *oscillation* of f at a .)

3. Let $A \subset \mathbb{R}^n$ be a closed rectangle, and let $f : A \rightarrow \mathbb{R}$ be a bounded function. If P and P' are any two partitions of A , show that

$$L(f, P) \leq U(f, P').$$

4. Let $f : A \rightarrow \mathbb{R}$ be integrable. Show that $|f|$ is integrable and that

$$\left| \int_A f \right| \leq \int_A |f|.$$

5. Let $A = [0, 1] \times [0, 1]$, and define $f : A \rightarrow \mathbb{R}$ as follows:

$$f(x, y) = \begin{cases} 0 & , \text{ if } x \notin \mathbb{Q} \\ 0 & , \text{ if } x \in \mathbb{Q} \text{ and } y \notin \mathbb{Q} \\ \frac{1}{q} & , \text{ if } x \in \mathbb{Q} \text{ and } y = \frac{p}{q} \text{ in lowest terms} \end{cases}$$

- (a) For each $\mathbf{a} \in A$, determine $o(f, \mathbf{a})$.
 - (b) Show that f is integrable on A . (What is $\int_A f$?)
6. Give an example of a closed set of measure zero that does not have content zero.
 7. Let $A = \{x \in [0, 1] \mid \text{the decimal expansion of } x \text{ has no 8's}\}$.
Let $B = \{n \in \mathbb{N} \mid \text{the decimal expansion of } n \text{ has no 8's}\}$.

- (a) Find the content of A .
- (b) Decide whether the infinite series $\sum_{n \in B} \frac{1}{n}$ converges or diverges.

8. For a function $f : [0, 1] \rightarrow \mathbb{R}$, let $A = \{x \in [0, 1] \mid f \text{ is not differentiable at } x\}$. Find such an f satisfying the following conditions:

- f is continuous.
- $f(0) = 0$
- $f(1) = 1$
- A has content zero.
- If $x \notin A$, then $f'(x) = 0$.

(Hint: Use the Cantor set.)