Math 23a, 2002.
Solution Set 8, Question 2-3.
Joshua Reyes

Question 2. Let a and b be real numbers. Using P to denote the set of positive elements, we define "less than" formally by the statement:

$$a < b \text{ iff } b - a \in P.$$

Use axioms P1-P8 for an ordered field and the definition of absolute value to show that:

(1) If $0 < a < b$, then $0 < a^2 < b^2$.
(2) If $a^2 < b^2$, then $|a| < |b|$.

Answer. Given $0 < a < b$, we know that $b - a \in P$. And since P is closed under addition and multiplication, we know that $(b + a)(b - a) \in P$, too. But if you expand that out you find that $b^2 - a^2 \in P$. Well, cool, 'cause that means $a^2 < b^2$. It's worth noting that $a^2 = a \cdot a$ and $b^2 = b \cdot b \in P$, too. Score.

For this second part, a lot of you proved like forty-seven cases, and don't get me wrong, that works. But some of you also attacked the contrapositive. That was easier to read, so I'll follow fold and do the like. Negate the hypotheses and conclusions and switch'em. In this case, pretend like $|b| \leq |a|$. If either a or $b = 0$, then quit. Now if $|a| = |b|$, then $a^2 = b^2$ for sure. So take $|a| > |b|$ and apply the above results. Then $a^2 > b^2$. Since $|a| \geq |b|$ implies $a^2 \geq b^2$, we get $a^2 < b^2$ implies $|a| < |b|$ for free.

Question 3. Let V be a vector space with an inner product $\langle \cdot, \cdot \rangle$ and an associated norm $\|\cdot\|$. We say that a linear transformation $A : V \to V$ is *norm-preserving* if $\|Av\| = \|v\|$ for every $v \in V$ and *inner product-preserving* if $\langle Au, Av \rangle = \langle u, v \rangle$ for all u, $v \in V$.

(a) Show that A is inner product-preserving if and only if it is norm-preserving. (Hint #1: One way is easy! Hint #2: Expand the identity $\|A(u + v)\|^2 = \|u + v\|^2$.)
(b) Let $V = \mathbb{R}^2$ with the usual inner product. Find all norm-preserving linear transformations/matrices.

Answer. ($\Rightarrow$) Going the first way:

$$\langle Av, Av \rangle = \langle v, v \rangle = \|v\|^2.$$

But if we forget that A is inner product-preserving, we'd be tempted to say $\langle Av, Av \rangle = \|Av\|^2$. And since both quantities are always at least zero, why not add $\|Av\| = \|v\|$ to the mix.

($\Leftarrow$) Now the other way: Since A is norm-preserving $\|A(v + u)\| = \|v + u\|$. We can certainly square both sides, so $\|A(v + u)\|^2 = \|v + u\|^2$. But that's the same thing as

$$\|v + u\|^2 = \langle v + u, v + u \rangle = \langle v, v \rangle + 2\langle v, u \rangle + \langle u, u \rangle$$
$$= \|v\|^2 + 2\langle v, u \rangle + \|u\|^2.$$

But we could've also said (always keeping in mind that A is linear)

$$\|A(v + u)\|^2 = \langle A(v + u), A(v + u) \rangle = \langle Av, Av \rangle + 2\langle Av, Au \rangle + \langle Au, Au \rangle$$
$$= \|Av\|^2 + 2\langle Au, Av \rangle + \|Au\|^2.$$

But oh ho! A is norm-preserving. Matching up terms we find that

$$2\langle Av, Au \rangle = 2\langle v, u \rangle.$$

That only works if A is inner product-preserving, too. Dude, those two conditions got to be interchangeable.

As for part (b), since A is norm-preseving, we just showed it's also got to respect the inner-product. That means orthonormal vectors get sent to another set of orthonormal vectors. After all the length stays the same and if $\langle v, u \rangle = 0$ then you betcha $\langle Av, Au \rangle$ better be zero, too. So, let e_1, e_2 be our friendly standard basis vectors. Then their images Ae_1 and Ae_2 are the columns of A. Let's pretend

$$A = \begin{matrix} a & b \\ c & d \end{matrix} .$$

Since the columns are normal $\sqrt{a^2 + c^2} = \sqrt{b^2 + d^2} = 1$. And 'cause they're orthogonal, $ac + bd = 0$. (p.s.– that's why these matrices are usually called *orthogonal.*) A lot of you correctly identified these matrices as the reflections and rotations. But if you did so without proof, it didn't count. Sorry.