MATH 23A SOLUTION SET #8 (PART D)

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Problem (6). *A set S is called a metric space if there exists a function $d : S \times S \to \mathbb{R}$ called the distance such that:*

(1) $d(x, y) = d(y, x)$, $\forall x, y \in S$
(2) $d(x, y) \geq 0$, $\forall x, y \in S$ *and* $d(x, y) = 0$ *iff* $x = y$
(3) $d(x, y) \leq d(x, z) + d(z, y)$, $\forall x, y, z \in S$

Show that the following are all instances of metric spaces:

(a). *S is a Euclidean space, with distance given by $d(u, v) = ||u - v||$.*

Solution. What this example shows is that given a norm in a vector space, we can always associate a metric to it, but the converse is not true - we can have a distance function on a vector space which does not come from a norm. An example of that is the discrete metric. (Why?) The only thing you needed to do to get full credit for this problem was to notice the correspondence between the properties of the norm given in problem 5 and the properties that a distance function is required to have.

Symmetry. For all $u, v \in S$ we have that:

$$d(u, v) = ||u - v|| = ||(-1)(v - u)|| = |-1|\,||v - u|| = ||v - u|| = d(v, u)$$

We have used the fact that the norm $||\cdot||$ on S must satisfy $||cv|| = |c|\,||v||$ for all $c \in \mathbb{R}$, $v \in S$, which is easy enough to check:

$$||cv|| = \sqrt{\langle cv, cv\rangle} = \sqrt{c^2\langle v, v\rangle} = |c|\sqrt{\langle v, v\rangle} = |c|\,||v||$$

Nonnegativity. For all $u, v \in S$ we have that:

$$d(u, v) = ||u - v|| = \sqrt{\langle u - v, u - v\rangle} \geq 0$$

If $u = v$, then $d(u, v) = ||u - v|| = ||0|| = \sqrt{\langle 0, 0\rangle} = 0$, and if $d(u, v) = 0$, then $\sqrt{\langle u - v, u - v\rangle} = 0$, and we know inner product is positive-definite, so that we must have $u - v = 0$.

Triangle Inequality. Recall a theorem (proven in lecture using the Cauchy-Schwarz inequality) that $||x+y|| \leq ||x||+||y||$ for all $x, y \in S$. Applying this theorem with $x = u-v$, $y = v - w$ we get the triangle inequality for d:

$$d(u, w) = ||u - w|| = ||(u - v) + (v - w)|| \leq ||u - v|| + ||v - w|| = d(u, v) + d(v, w)$$

□

(b). *S is any set, and the discrete metric on it is defined as follows:*

$$d(x, y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$$

Solution. Even though the discrete metric is not terribly interesting, it is important to keep it in mind, as it might often serve you well as a pathological counterexample that will disprove some claims that intuitively seem true but are actually false. Proving this is a metric was not hard, and most people got it right - here is how it might look:

Symmetry. If $x = y$, then $d(x, y) = d(y, x) = 0$. If $x \neq y$, then $d(x, y) = d(y, x) = 1$.

Nonnegativity. Clear from the definition.

Triangle Inequality. One way to show the triangle inequality holds is to check all possible cases of equality between $x, y, z \in S$. It turns out there are 5 of those, namely: $x = y = z$, $x = y \neq z$, $x = z \neq y$, $y = z \neq x$ and $x \neq y \neq z \neq x$, and it is easy to check all 5 cases. Another way to prove that $d(x, y) \leq d(x, z) + d(z, y)$ is to notice that $d(x, y) \in \{0, 1\}$ and $d(x, z) + d(z, y) \in \{0, 1, 2\}$, so it is enough to check that $d(x, y) = 1$ and $d(x, z) + d(z, y) = 0$ cannot both be true. But it is clear that, if $d(x, z) + d(z, y) = 0$, then $x = z$ and $z = y$, so that $x = y$ and $d(x, y) = 0 \neq 1$. □

(c). *Assuming S is already a metric space with distance d, define new distance d' on S as follows:*

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

Solution. Notice that for any $x, y \in S$, $d'(x, y) < 1$, so that with respect to the distance function d', S is bounded. It follows that any set can be bounded if you pick the right way to measure distance! Proving that d' is a distance function relies on the corresponding properties of d, and only the triangle inequality was somewhat tricky.

Symmetry. Since d is a distance on S, we know that $d(x, y) = d(y, x)$ for all $x, y \in S$, so that:

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)} = \frac{d(y, x)}{1 + d(y, x)} = d'(y, x)$$

Nonnegativity. Since $d(x, y) \geq 0$ for all $x, y \in S$, we have $1 + d(x, y) > 0$ as well, so that $d'(x, y) = \frac{d(x,y)}{1+d(x,y)} \geq 0$. If $x = y$, then $d(x, y) = 0$, so that $d'(x, y) = 0$ as well. If $d'(x, y) = 0$, then $d(x, y) = 0$, and since d is a distance function, we must have $x = y$.

Triangle Inequality. Fix $x, y, z \in S$ and let $a = d(x, y)$, $b = d(x, z)$, $c = d(z, y)$. Since d is a distance function on S, we know that $a, b, c \geq 0$ and $a \leq b + c$. We want to show that $d'(x, y) \leq d'(x, z) + d'(z, y)$, which in this notation is:

$$\frac{a}{1 + a} \leq \frac{b}{1 + b} + \frac{c}{1 + c}$$

$$\Leftrightarrow a(1 + b)(1 + c) \leq b(1 + a)(1 + c) + c(1 + a)(1 + b)$$

$$\Leftrightarrow a + ab + ac + abc \leq b + c + ab + ac + 2bc + 2abc$$

$$\Leftrightarrow a \leq b + c + 2bc + abc$$

But we know that $a \leq b + c$ and that $0 \leq 2bc + abc$, so the above inequality holds and $d'(x, y) \leq d'(x, z) + d'(z, y)$. □

(d). *$S = \mathbb{R}^2$, and the distance d is defined as follows:*

$$d((a, b), (c, e)) = \begin{cases} 0, & (a, b) = (c, e) \\ \sqrt{a^2 + b^2} + \sqrt{c^2 + e^2}, & (a, b) \neq (c, e) \end{cases}$$

Solution. Let $v = (a, b)$, $w = (c, e)$, $u = (f, g) \in \mathbb{R}^2$

Symmetry. If $v = w$, then $d(v, w) = 0 = d(w, v)$. If $v \neq w$, then:

$$d(v, w) = \sqrt{a^2 + b^2} + \sqrt{c^2 + e^2} = \sqrt{c^2 + e^2} + \sqrt{a^2 + b^2} = d(w, v)$$

Nonnegativity. It is clear from the definition that $d(v, w) \geq 0$ for all $v, w \in]rr^2$ and that if $v = w$ then $d(v, w) = 0$. It is left to check that $d(v, w) = 0$ implies $v = w$. But if we have $\sqrt{a^2 + b^2} + \sqrt{c^2 + e^2} = 0$, we must have $a = b = c = e = 0$ so that $v = w = 0$.

Triangle Inequality. In proving that $d(v, w) \leq d(v, u) + d(u, w)$ we can assume $u \neq v \neq w \neq u$ since the statement is otherwise trivial. Even in this case, the triangle inequality is not much harder to check:

$$d(v, u)+d(u, w) = \sqrt{a^2 + b^2}+\sqrt{f^2 + g^2}+\sqrt{f^2 + g^2}+\sqrt{c^2 + e^2} \geq \sqrt{a^2 + b^2}+\sqrt{c^2 + e^2} = d(v, w)$$

□