MATH 23a, FALL 2002
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS
(Final Version) Homework Assignment # 9
Due: December 6, 2002

1. Read Sections 16-18 of Chapter 5 in Curtis.

2. (Not required) Let V be a vector space over F. Consider the k-linear forms $f : V^k \longrightarrow F$. For any two such forms f_1 and f_2 and any scalar $c \in F$, we define:

$$(f_1 + f_2)(\mathbf{v}_1, \mathbf{v}_2) = f_1(\mathbf{v}_1, \mathbf{v}_2) + f_2(\mathbf{v}_1, \mathbf{v}_2), \qquad \forall \mathbf{v}_1, \mathbf{v}_2 \in V$$

$$(cf_1)(\mathbf{v}_1, \mathbf{v}_2) = c \cdot f_1(\mathbf{v}_1, \mathbf{v}_2), \qquad \forall \mathbf{v}_1, \mathbf{v}_2 \in V$$

Convince yourself that the collection of k-linear forms on V form a vector space over F with addition and scalar multiplication defined as above.

3. (A) Show that not every skew-symmetric multilinear form $f : V^n \longrightarrow F$ is alternating by constructing an example. (Note that the only cases where this can happen are over fields F wherein $1 + 1 = 0$. Follow Halmos' reasoning in the proof that alternating implies skew-symmetric, Section 30, Theorem 1.)

4. (B) Let $dim(V) = n$, and let $f : V^k \longrightarrow F$ be an alternating k-linear form with $k < n$. Show by example that it is possible to have a set of k linearly independent vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ in V such that $f(\mathbf{v}_1, \dots, \mathbf{v}_k) = 0$. (Make sure that $k \geq 2$ so that f can be alternating!)

5. (C) In the following, we consider vector spaces of bilinear forms:
   (a) Let $V = \mathbb{R}^2$. Show that the form $f : V^2 \longrightarrow \mathbb{R}$ defined by $f((a, b), (c, d)) = ad - bc$ is bilinear and alternating.
   (b) Now let $V = \mathbb{R}^3$. Construct two *linearly independent* alternating bilinear forms $f : V^2 \longrightarrow \mathbb{R}$.
   (c) Determine the dimensions of the spaces of alternating bilinear forms on $V = \mathbb{R}^2$ and $V = \mathbb{R}^3$.

6. (deferred) Recall the definition of the *transpose* of a matrix, as referred to in homework problem #8.8, and prove the following:

   Theorem: If A is an $n \times n$ matrix, then $det(A^t) = det(A)$.

7. (D) Let $V = \mathbb{R}^n$, and let $\mathbf{u}, \mathbf{v} \in V$. If $A : V \longrightarrow V$ is a linear transformation, then define the following bilinear form:

$$f_A(\mathbf{u}, \mathbf{v}) = \mathbf{u}^t A \mathbf{v}$$

(a) (Not required) Convince yourself that f_A is indeed a bilinear form.

(b) Give a necessary and sufficient condition on A that makes f_A an inner product. (Full points for a complete answer in the $n = 2$ case.)

(c) Give a necessary and sufficient condition on A that makes f_A alternating.

(Hint #1: You might consider the $n = 2$ case first to get a feel for this bilinear form. Hint #2: Your answers just might involve the transpose!)

8. (deferred) Let $A : V \longrightarrow V$ be a linear transformation on a finite-dimensional vector space, and by slight abuse of notation, let A also be the matrix for this transformation with respect to a fixed basis. Using the following method, we determine the eigenvalues of A:

$$\begin{aligned} \lambda \text{ is an eigenvalue for } A &\iff V_\lambda = Ker(A - \lambda I) \text{ is non-trivial} \\ &\iff A - \lambda I \text{ is not invertible} \\ &\iff det(A - \lambda I) = 0 \end{aligned}$$

Thus we are inspired to make the following definition:

$p_A(\lambda) = det(A - \lambda I)$ is called the **characteristic polynomial** of A

The eigenvalues of A will be the roots of the characteristic polynomial.

(a) Prove that no scalar $\lambda_0 \in F$ is an eigenvalue for A unless it is a root of $p_A(\lambda)$.

(b) If $p_A(\lambda) = (\lambda - \lambda_0)^k \cdot q(\lambda)$ with $q(\lambda_0) \neq 0$, then we say that the eigenvalue λ_0 has *algebraic multiplicity* equal to k. (That is, λ_0 is a root of $p_A(\lambda)$ of order k.) Show that the geometric multiplicity (which, by definition, is the dimension of the corresponding eigenspace) of an eigenvalue is less than or equal to its algebraic multiplicity.

(c) Use this method to find all eigenvalues of the real matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Is the matrix A diagonalizable? Explain.

9. (deferred) Show that $A : V \longrightarrow V$ is invertible if and only if $det(A) \neq 0$. (We have used this fact several times already, including in problem #2. The point of this exercise is to make you think carefully about the steps we used when we made the transition from alternating forms to determinants.)