

MATH 23b, SPRING 2002  
THEORETICAL LINEAR ALGEBRA  
AND MULTIVARIABLE CALCULUS  
Homework Assignment # 9  
Due: April 26, 2002

Homework Assignment #9 (Version 2)

1. Read Edwards Sections 4.5–4.6 and 5.1, especially pp. 295–301.
2. In the following problem, we compute the volume of the  $n$ -dimensional unit ball,  $B^n \subset \mathbb{R}^n$ , in the even and odd cases, respectively, to be:

$$v(B^{2m}) = \frac{\pi^m}{m!} \quad \text{and} \quad v(B^{2m+1}) = \frac{2^{m+1}\pi^m}{1 \cdot 3 \cdot 5 \cdots (2m+1)}$$

- (a) Read Edwards' problem 5.17 (p. 267) and problem 1.8 (p. 213) for one solution. Note that Edwards uses the notation  $I_n$  to denote the value of the integral  $\int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta$ .
- (b) Do Edwards' problem 5.18 (p. 267):  
Let  $B^2 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\}$ .  
Let  $Q = \{(x_3, \dots, x_n) \in \mathbb{R}^{n-2} \mid |x_i| \leq 1, \forall i\}$ . Then  $B^n \subset B^2 \times Q$ .  
Let  $\varphi : B^2 \times Q \rightarrow \mathbb{R}$  be the characteristic function of  $B^n$ . Then

$$v(B^n) = \int_{B^2} \left( \int_Q \varphi(x_1, \dots, x_n) \, dx_3 \dots dx_n \right) dx_1 dx_2.$$

If  $(x_1, x_2) \in B^2$  is fixed, then  $\varphi$ , considered as a function of the variables  $x_3, \dots, x_n$  is the characteristic function of  $B_r^{n-2}$ , the  $(n-2)$ -ball of radius  $r = \sqrt{1 - x_1^2 - x_2^2}$ . Hence

$$\int_Q \varphi(x_1, \dots, x_n) \, dx_3 \dots dx_n = (1 - x_1^2 - x_2^2)^{(n-2)/2} \cdot v(B^{n-2}).$$

Now, introduce polar coordinates on  $\mathbb{R}^2$  to show that

$$\int_{B^2} (1 - x_1^2 - x_2^2)^{(n-2)/2} \, dx_1 dx_2 = \frac{2\pi}{n}.$$

Conclude that  $v(B^n) = \frac{2\pi}{n} \cdot v(B^{n-2})$ , and use induction and the base cases (that is, that  $v(B^2) = \pi$  and  $v(B^3) = \frac{4}{3}\pi$ ) to prove the given formulas for  $v(B^n)$ .

3. Adapted from Edwards' problem 5.19:

- (a) Three-dimensional Euclidean space can be represented via the standard spherical coordinates transformation:

$$(x, y, z) = T(\rho, \theta, \varphi) = (\rho \sin \varphi \cos \theta, \rho \sin \varphi \sin \theta, \rho \cos \varphi),$$

where  $\rho$  is radius of the sphere on which  $(x, y, z)$  lies (the distance from the point to the origin),  $\theta$  is the angle  $(x, y, 0)$  makes with the  $x$ -axis (the longitude), and  $\varphi$  is the angle  $(x, y, z)$  makes with the  $z$ -axis (the latitude).

Compute  $|\det JT|$  as a function of  $\rho$ ,  $\theta$ , and  $\varphi$ .

- (b) More generally,  $n$ -dimensional spherical coordinates are given by the map  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  given as:

$$\begin{aligned} x_1 &= \rho \cos \varphi_1 \\ x_2 &= \rho \sin \varphi_1 \cos \varphi_2 \\ x_3 &= \rho \sin \varphi_1 \sin \varphi_2 \cos \varphi_3 \\ &\vdots \\ x_{n-1} &= \rho \sin \varphi_1 \cdots \sin \varphi_{n-2} \cos \theta \\ x_n &= \rho \sin \varphi_1 \cdots \sin \varphi_{n-2} \sin \theta \end{aligned}$$

Show by induction that

$$|\det JT| = \rho^{n-1} \sin^{n-2} \varphi_1 \sin^{n-3} \varphi_2 \cdots \sin^2 \varphi_{n-3} \sin \varphi_{n-2}.$$

- (c) Let  $B^3 = \{(x, y, z) | x^2 + y^2 + z^2 \leq 1\}$  be the unit ball in  $\mathbb{R}^3$ , and let  $f(x, y, z) = e^{(x^2+y^2+z^2)^{3/2}}$ . Use the spherical coordinates change of variables to compute the integral  $\int_{B^3} f$ .

4. Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be positive and continuous, and suppose that

$$\int \int_D f = \int_0^1 \left( \int_y^{\sqrt{2-y^2}} f(x, y) dx \right) dy.$$

Sketch the region  $D$  and interchange the order of integration.

5. Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be continuous, and let  $B_\varepsilon$  be the ball of radius  $\varepsilon$  centered at the point  $\mathbf{x} \in \mathbb{R}^n$ . Show that:

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{v(B_\varepsilon)} \int_{B_\varepsilon} f = f(\mathbf{x}).$$

6. Problem 5.7 from p. 264 of Edwards. Use the change of variables  $u = x - y$  and  $v = x + y$  to evaluate the integral  $\int \int_D e^{(x-y)/(x+y)} dx dy$ , where  $D$  is the region in  $\mathbb{R}^2$  bounded by the axes  $x = 0$  and  $y = 0$  and the line  $x + y = 1$ .