If you don't understand anything about any of the solutions here, or if you spot mistakes, feel free to e-mail me.

If you didn't do well on this section, don't worry. Part (b) was difficult, and very few people received 10/10.

7b

We wish to find necessary and sufficient conditions such that $f(u, v)$ is an inner-product.

Bilinear. We are given that $f(u, v)$ is bilinear for any A.

Symmetric. First, note that $f(u, v)$ is a scalar, so it is equal to its own transpose.

$$f(u, v) = f(v, u) \Leftrightarrow u^t A v = v^t A u$$
$$\Leftrightarrow (u^t A v)^t = v^t A u$$
$$\Leftrightarrow v^t A^t u = v^t A u.$$

At this point, if $A = \{a_{ij}\}$, if we let $v = e_j$ and $u = e_i$, then $v^t A u = a_{ij}$, which implies that $A = A^t$. These steps are reversible, so this condition is both necessary and sufficient for f being symmetric.

Positive Definite. This is the tricky one. We already know $A = A^t$, so let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$. Let $u = (1, 0)$, then $f(u, u) = a$, which implies $a > 0$. Similarly, if $u = (0, 1)$, we find $d > 0$.

If $u = (0, 0)$, then $f(u, u) = 0$, so this property implies no extra conditions.

Given an inner product, every vector space has an orthogonal basis. (This is a direct consequence of Grahm-Shmidt Orthonormalization.) Let $v_1 = (1, 0)$. A basis vector orthogonal to b_1 will have the form $v_2 = (x, 1)$. Now, $f(v_1, v_2) = ax + b = 0$, so $x = -b/a$. Now, $f(v_2, v_2) = -b^2/a + d$. For $-b^2/a + d > 0$, we must have $-\frac{b^2}{a} > -d$ and $b^2 < ad$.

Finally, we need to note that $b^2 < ad$ is sufficient. Here we use the orthogonality of our basis vectors and bilinearity. If $f(v_1, v_1) > 0$, $f(v_2, v_2) > 0$ (both of which we've checked) and $f(v_1, v_2) = 0$ (we constructed it this way), then $f(u, v)$ is positive definite on span $\{v_1, v_2\}$.

To repeat, the necessary and sufficient conditions are:

$$\boxed{a > 0, d > 0, b = c, b^2 < ad}$$

Notes:

1. Some people claimed that they were constructing an orthonormal basis. Orthonormal means orthogonal and normalized ($\langle v, v, = \rangle 1$)—this does not have to be the case for e_1.

2. Many people solved a quadratic to find the $b^2 < ad$ condition. If you did this, you had to explain why this condition was sufficient. The quadratic formula wasn't quite enough, because you used the quadratic formula setting one side of the equation to zero.

7c

We want $f(u, v)$ to be alternating. If we expand by bilinearity, this implies that $f(u, v) = -f(v, u)$. Now we can use an argument similar to the argument we used for symmetry above. For all u, v:

$$f(u, v) = -f(v, u) \Leftrightarrow u^t A v = -v^t A u$$
$$\Leftrightarrow (u^t A v)^t = -v^t A u$$
$$\Leftrightarrow v^t A^t u = -v^t A u$$
$$\Leftrightarrow A^t = -A.$$

The proof is bidirectional, so the last condition is both necessary and sufficient. Note that this condition is the same as saying if $A = \{a_{ij}\}$, $a_{ij} = -a_{ji}$, which also happens to imply that the diagonal is zero.

Most people did this by picking a basis and proving $a_{ij} = -a_{ji}$. That's fine, but the above is nifty.

You were supposed to prove this problem in the general case.